

Graph Theory

Part Three

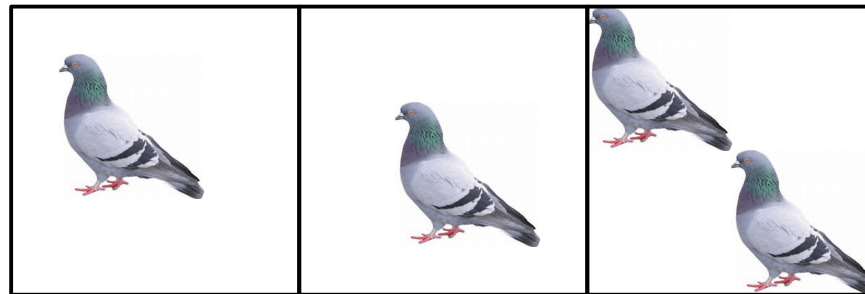
Outline for Today

- ***The Pigeonhole Principle***
 - A simple yet surprisingly effective fact.
- ***Graph Theory Party Tricks***
 - Cool tricks to try at your next group meeting.
- ***A Little Movie Puzzle***
 - Who watched what?

The Pigeonhole Principle

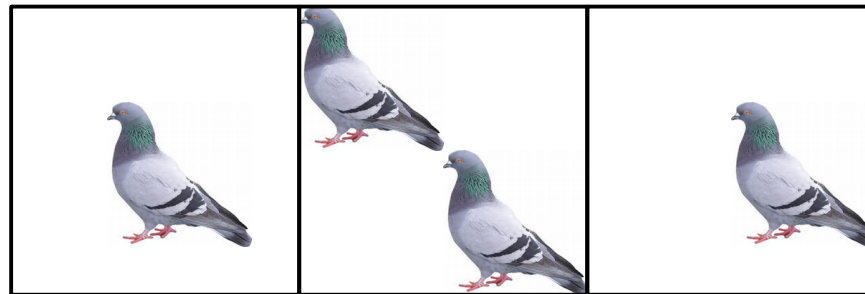
The Pigeonhole Principle

- ***Theorem (The Pigeonhole Principle):***
If m objects are distributed into n bins and $m > n$, then at least one bin will contain at least two objects.



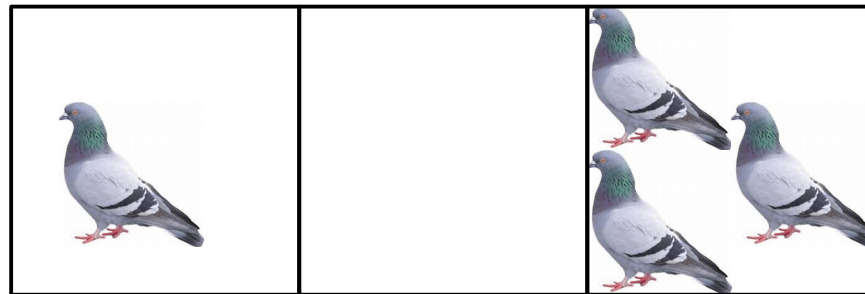
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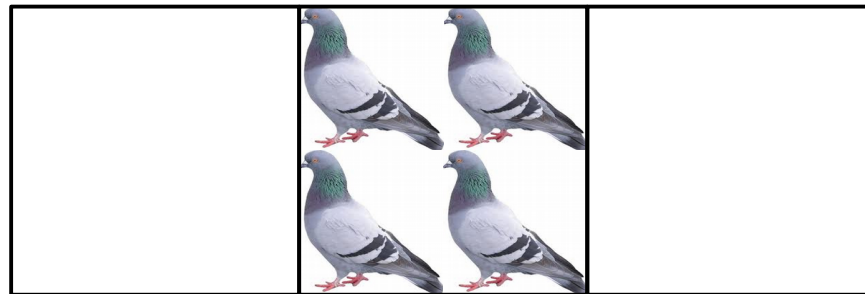
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NO MORE
- PIGEON HOLES?!



$$m = 4, n = 3$$

Some Simple Applications

- Any group of 367 people must have a pair of people that share a birthday.
 - 366 possible birthdays (pigeonholes).
 - 367 people (pigeons).
- Two people in San Francisco have the exact same number of hairs on their head.
 - Maximum number of hairs ever found on a human head is no greater than 500,000.
 - There are over 800,000 people in San Francisco.

Theorem (The Pigeonhole Principle): If m objects are distributed into n bins and $m > n$, then at least one bin will contain at least two objects.

Let A and B be finite sets (sets whose cardinalities are natural numbers) and assume $|A| > |B|$. How many of the following statements are true?

- (1) If $f : A \rightarrow B$, then f is injective.
- (2) If $f : A \rightarrow B$, then f is not injective.
- (3) If $f : A \rightarrow B$, then f is surjective.
- (4) If $f : A \rightarrow B$, then f is not surjective.

Answer at

<https://pollev.com/cs103>

Proving the Pigeonhole Principle

Theorem: If m objects are distributed into n bins and $m > n$, then there must be some bin that contains at least two objects.

Proof: Suppose for the sake of contradiction that, for some m and n where $m > n$, there is a way to distribute m objects into n bins such that each bin contains at most one object.

Number the bins $1, 2, 3, \dots, n$ and let x_i denote the number of objects in bin i . There are m objects in total, so we know that

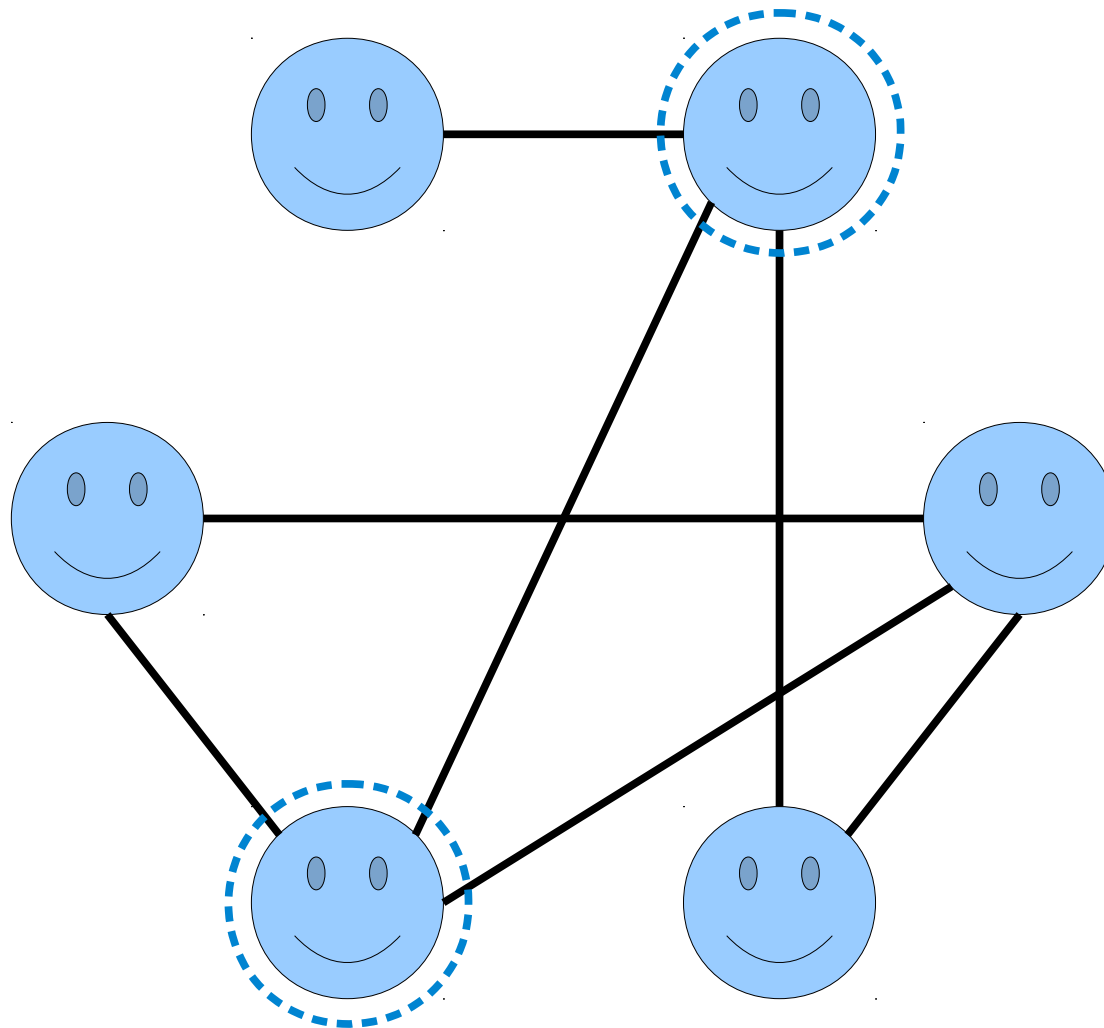
$$m = x_1 + x_2 + \dots + x_n.$$

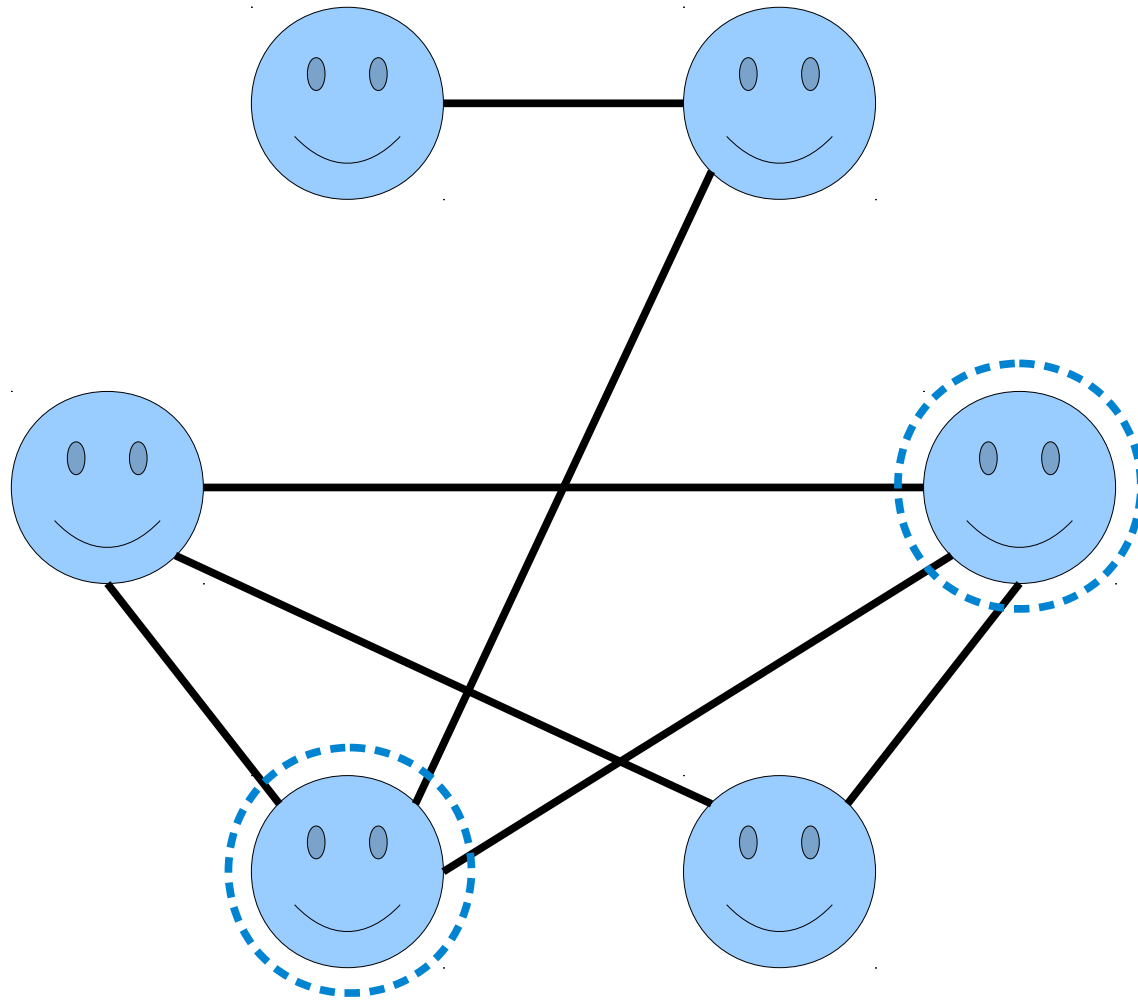
Since each bin has at most one object in it, we know $x_i \leq 1$ for each i . This means that

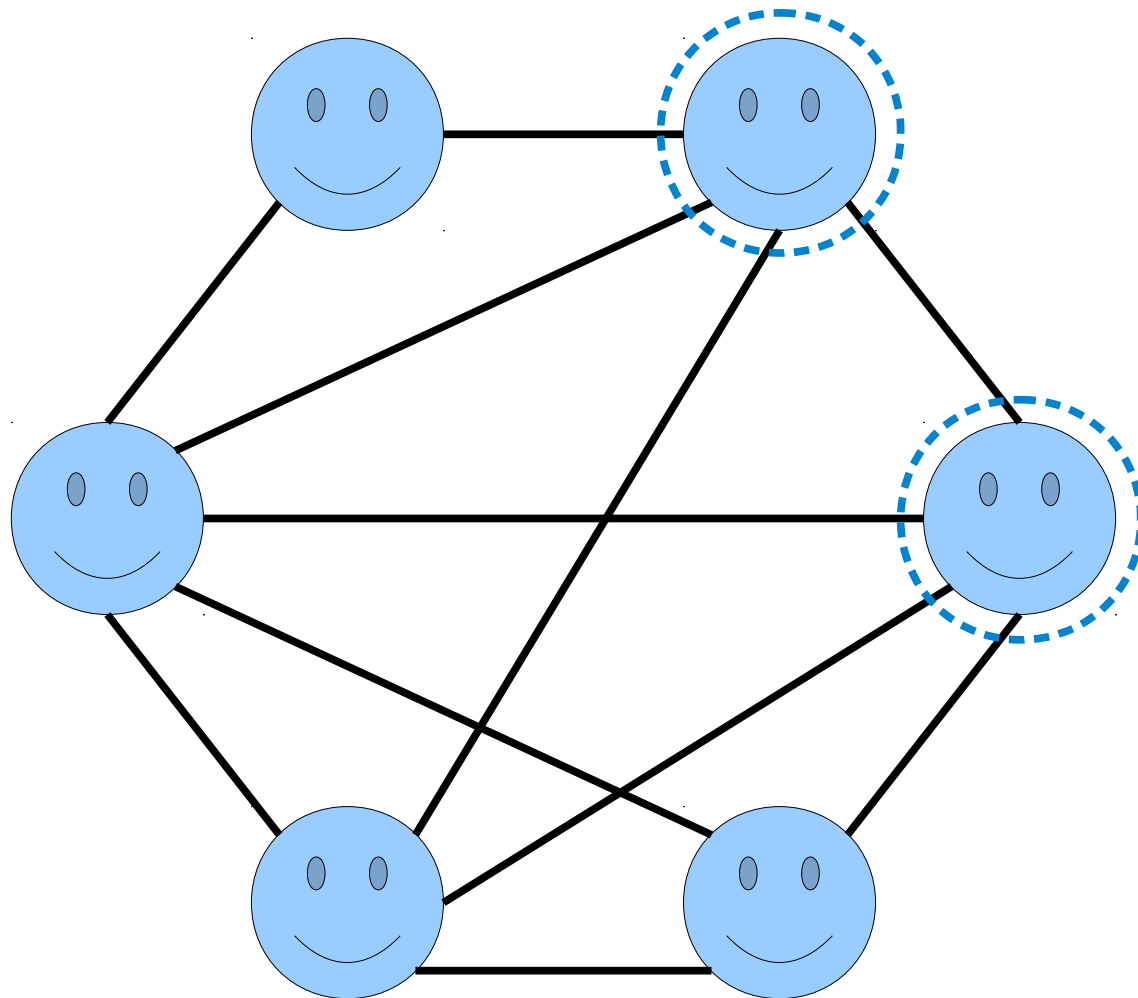
$$\begin{aligned} m &= x_1 + x_2 + \dots + x_n \\ &\leq 1 + 1 + \dots + 1 \quad (n \text{ times}) \\ &= n. \end{aligned}$$

This means that $m \leq n$, contradicting that $m > n$. We've reached a contradiction, so our assumption must have been wrong. Therefore, if m objects are distributed into n bins with $m > n$, some bin must contain at least two objects. ■

Pigeonhole Principle Party Tricks

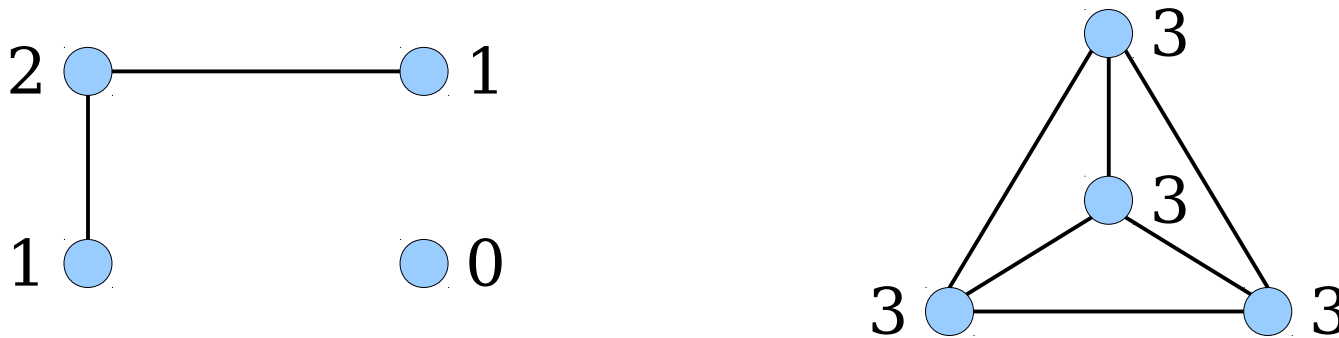




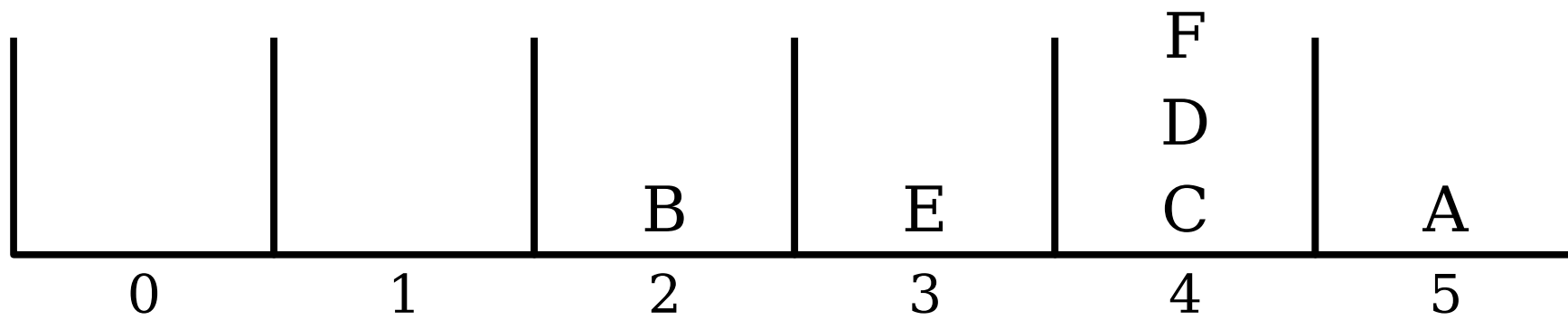
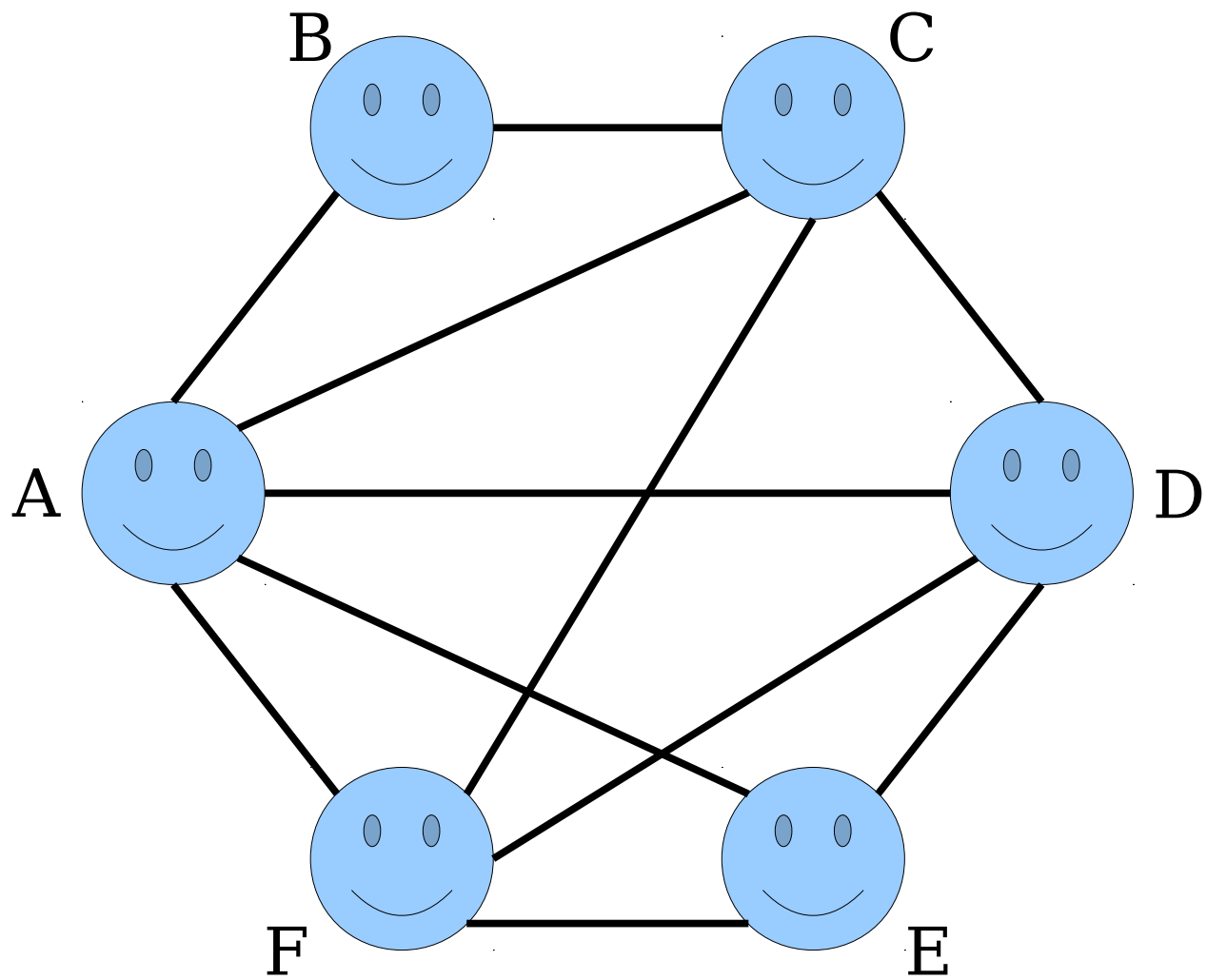


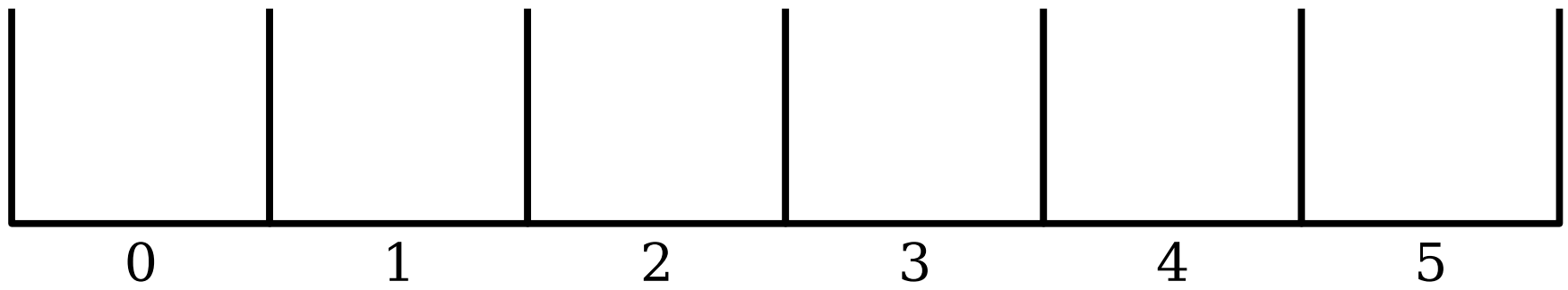
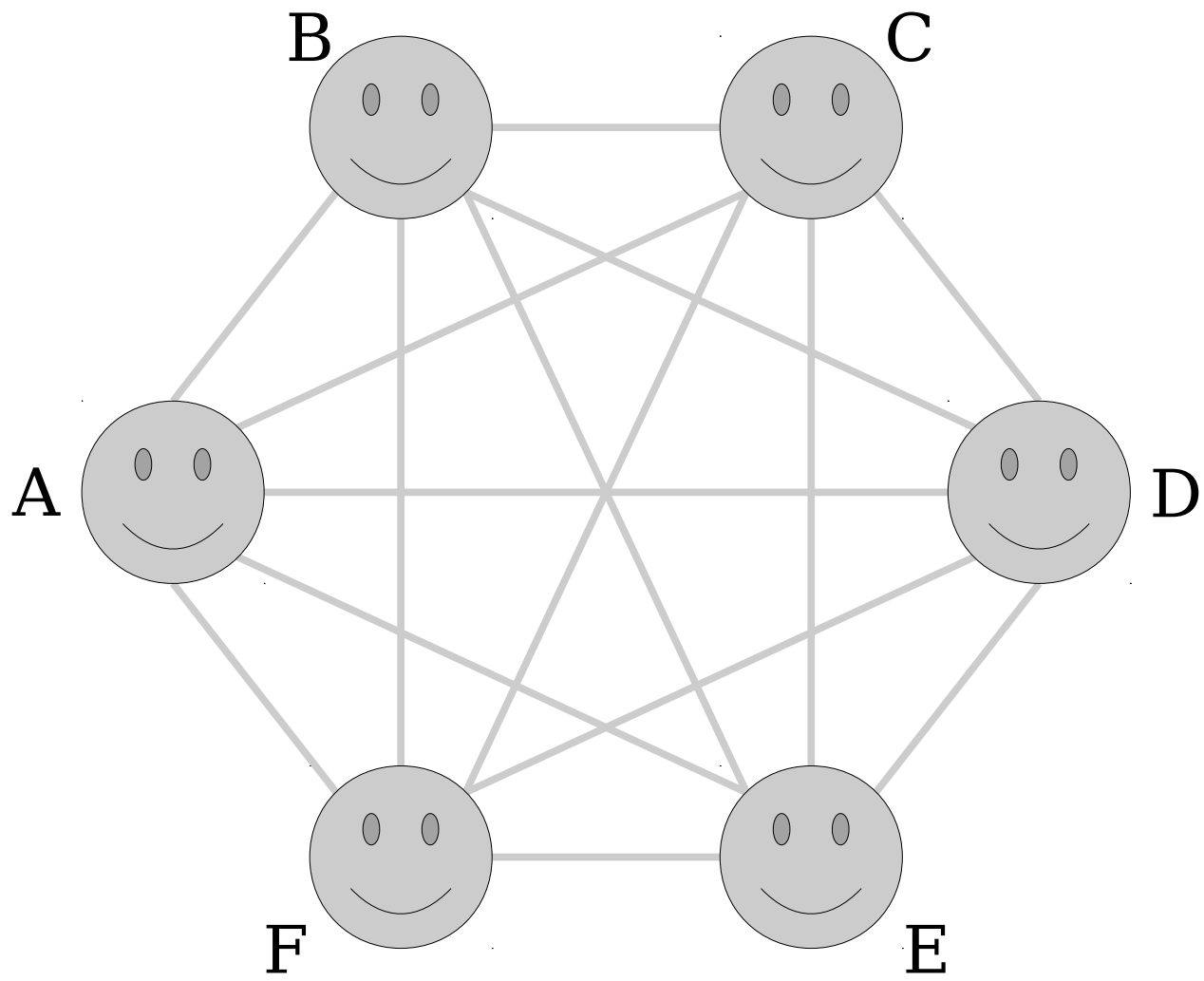
Degrees

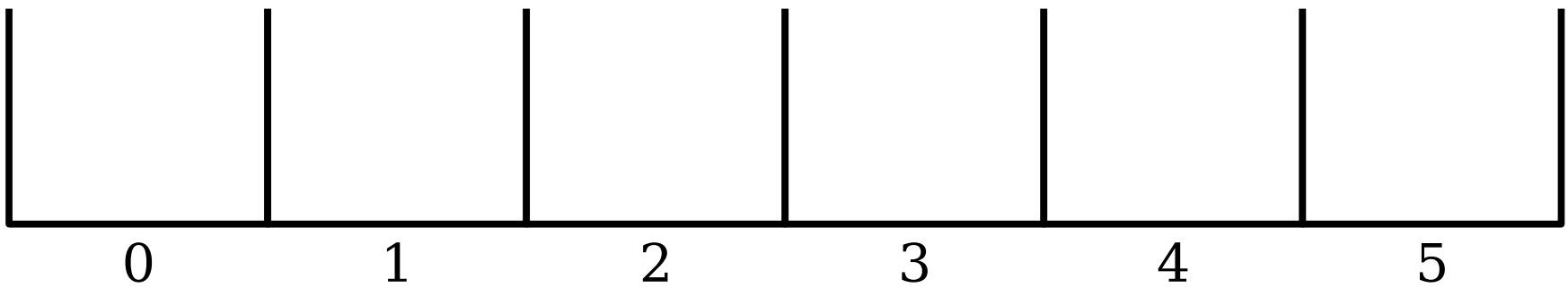
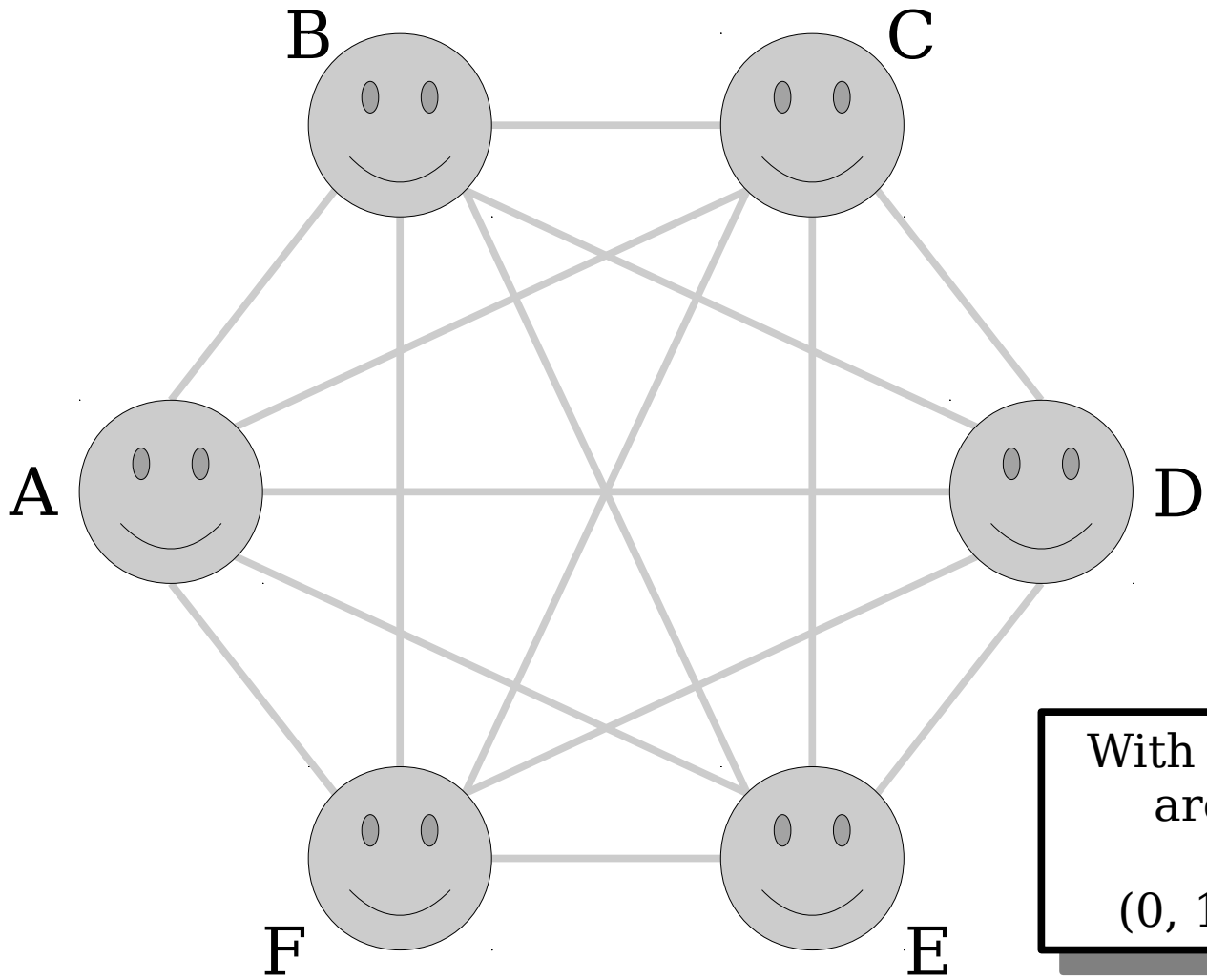
- The **degree** of a node v in a graph is the number of nodes that v is adjacent to.

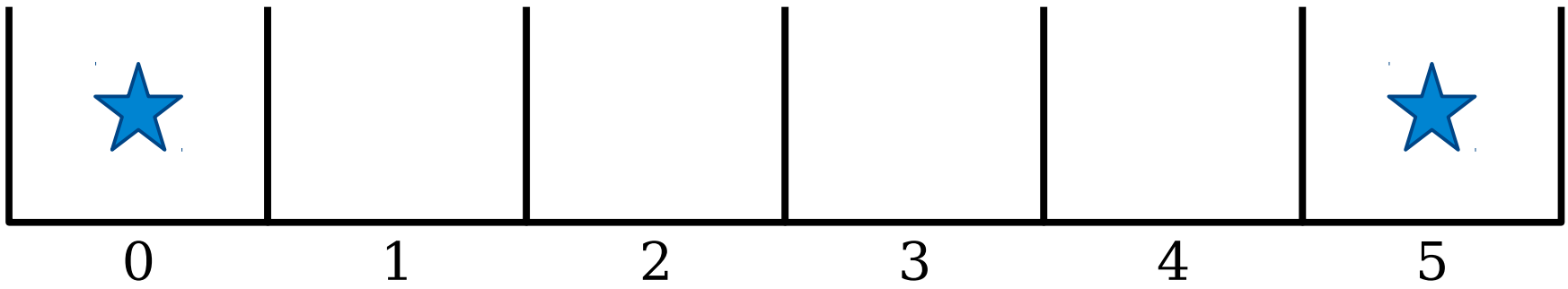
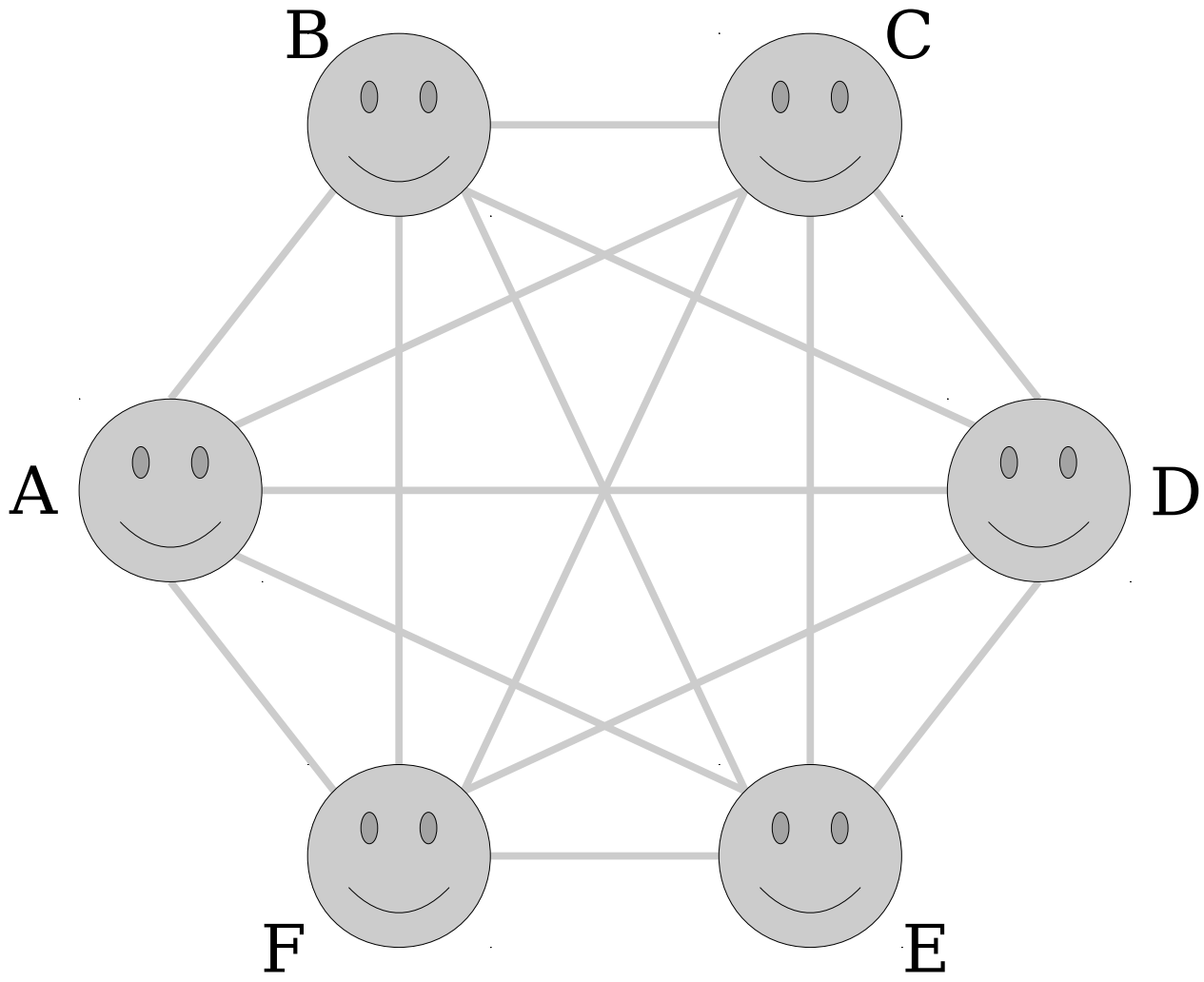


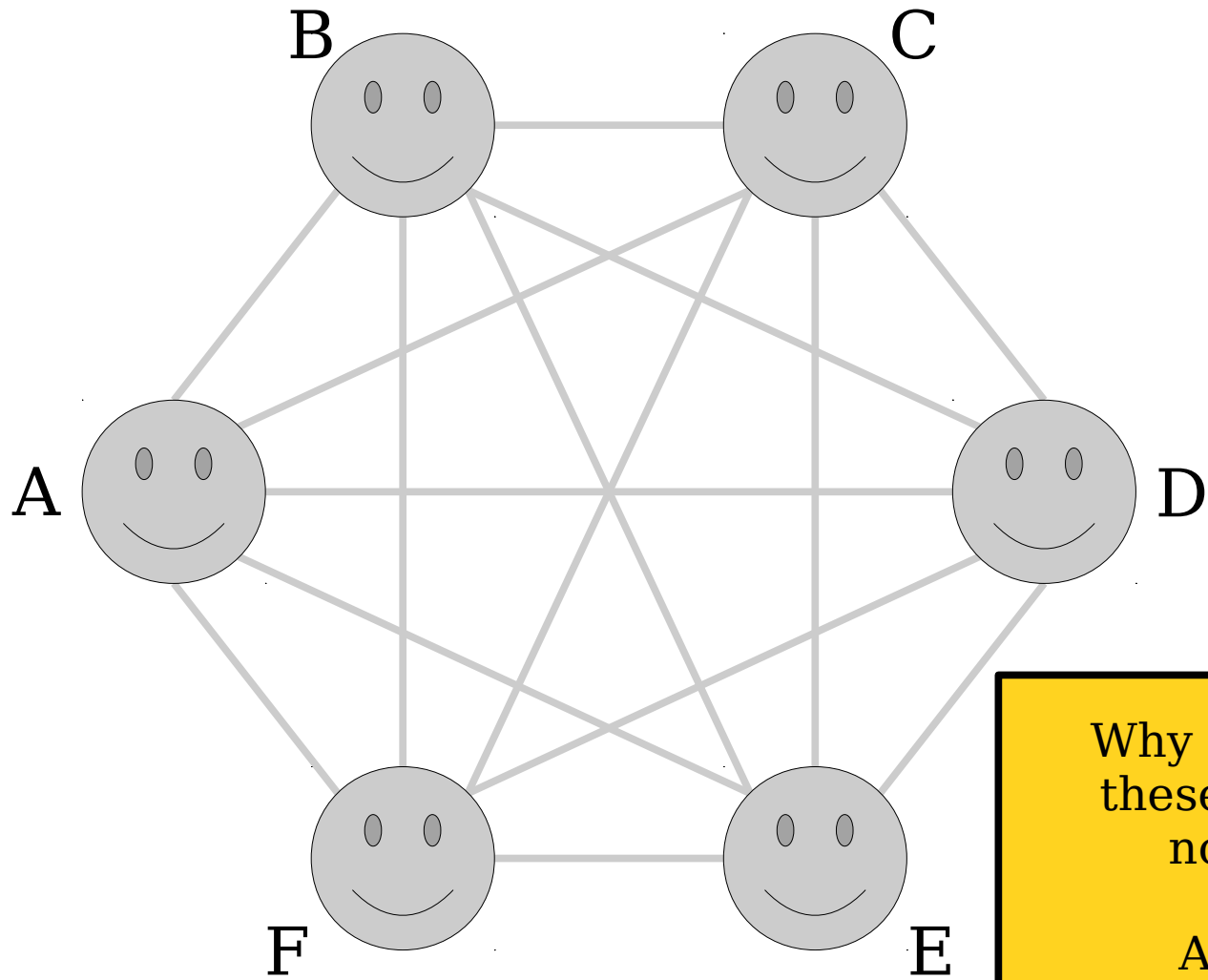
- Theorem:** Every graph with at least two nodes has at least two nodes with the same degree.
 - Equivalently: at any party with at least two people, there are at least two people with the same number of friends at the party.





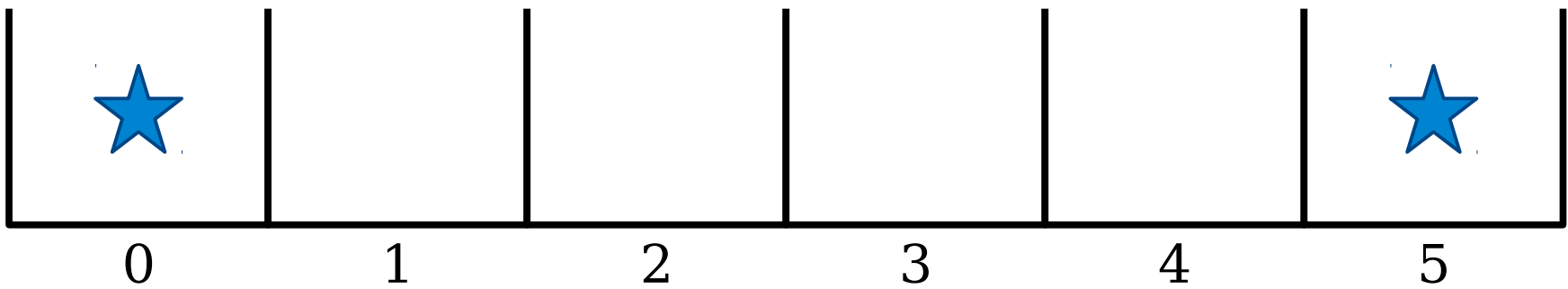


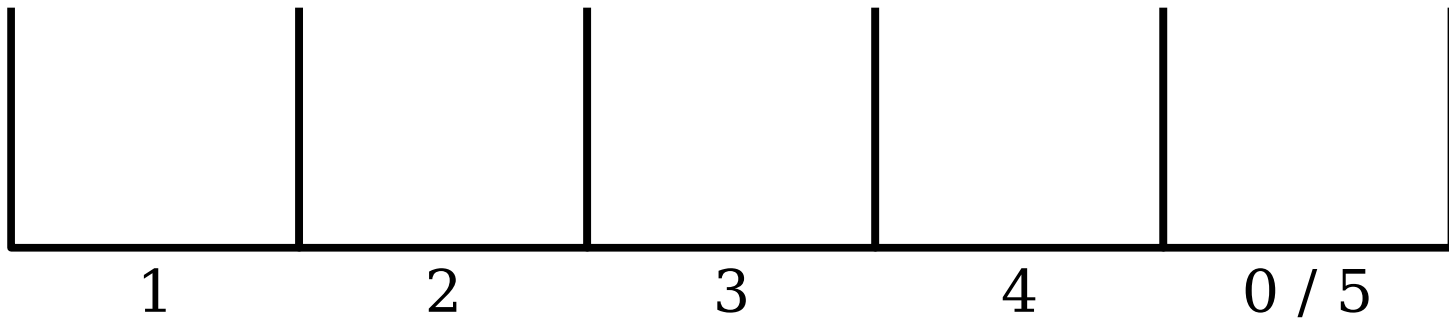
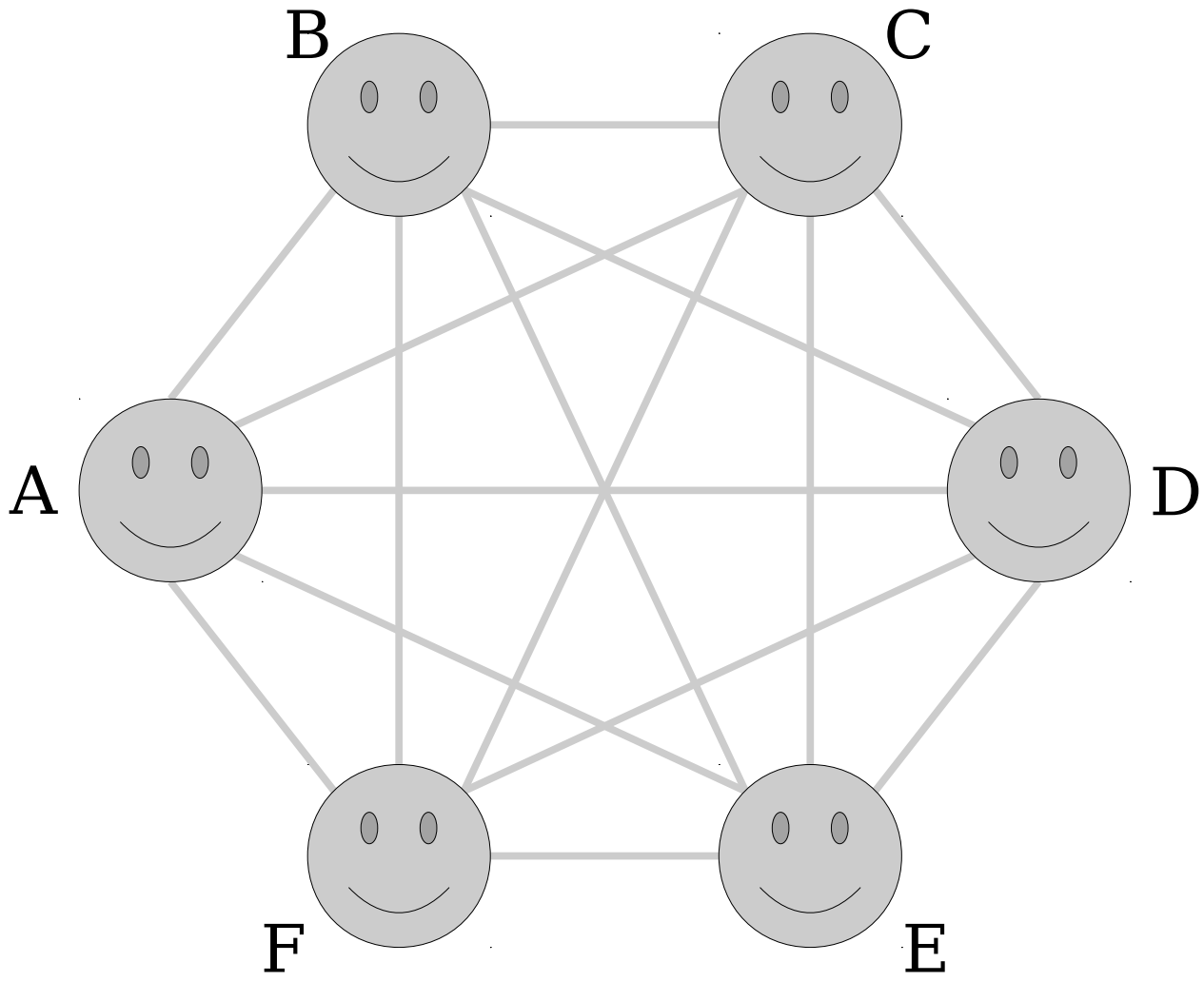




Why can't both of these buckets be nonempty?

Answer at
pollev.com/cs103





Theorem: In any graph with at least two nodes, there are at least two nodes of the same degree.

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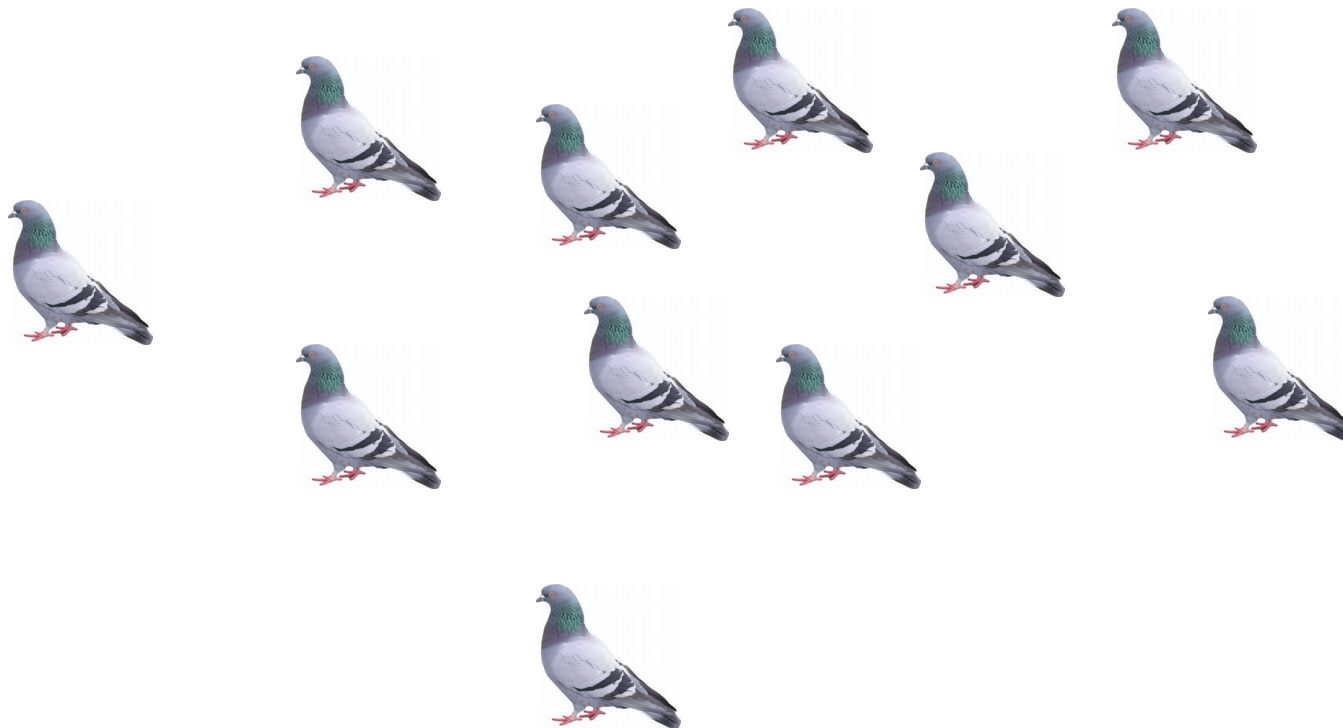
Theorem: In any graph with at least two nodes, there are at least two nodes of the same degree.

Proof 2: Assume for the sake of contradiction that there is a graph G with $n \geq 2$ nodes where no two nodes have the same degree. There are n possible choices for the degrees of nodes in G , namely $0, 1, 2, \dots, n - 1$, so this means that G must have exactly one node of each degree. However, this means that G has a node of degree 0 and a node of degree $n - 1$. (These can't be the same node, since $n \geq 2$.) This first node is adjacent to no other nodes, but this second node is adjacent to every other node, which is impossible.

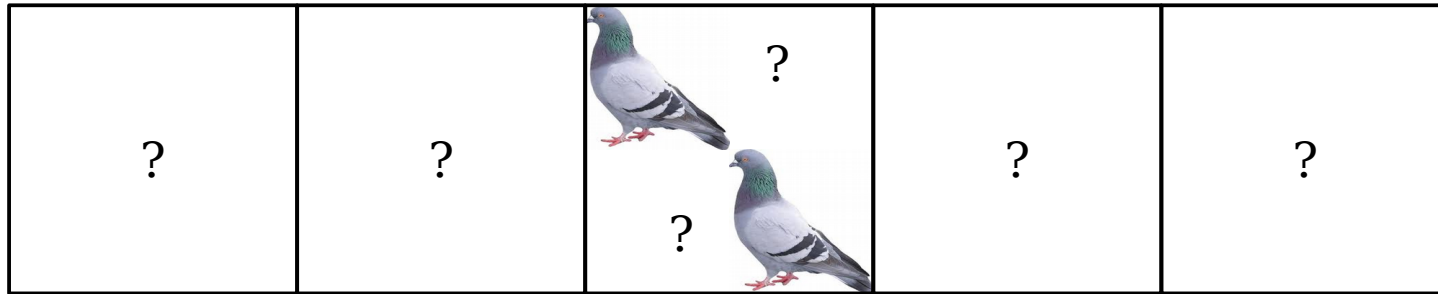
We have reached a contradiction, so our assumption must have been wrong. Thus if G is a graph with at least two nodes, G must have at least two nodes of the same degree. ■

The Generalized Pigeonhole Principle

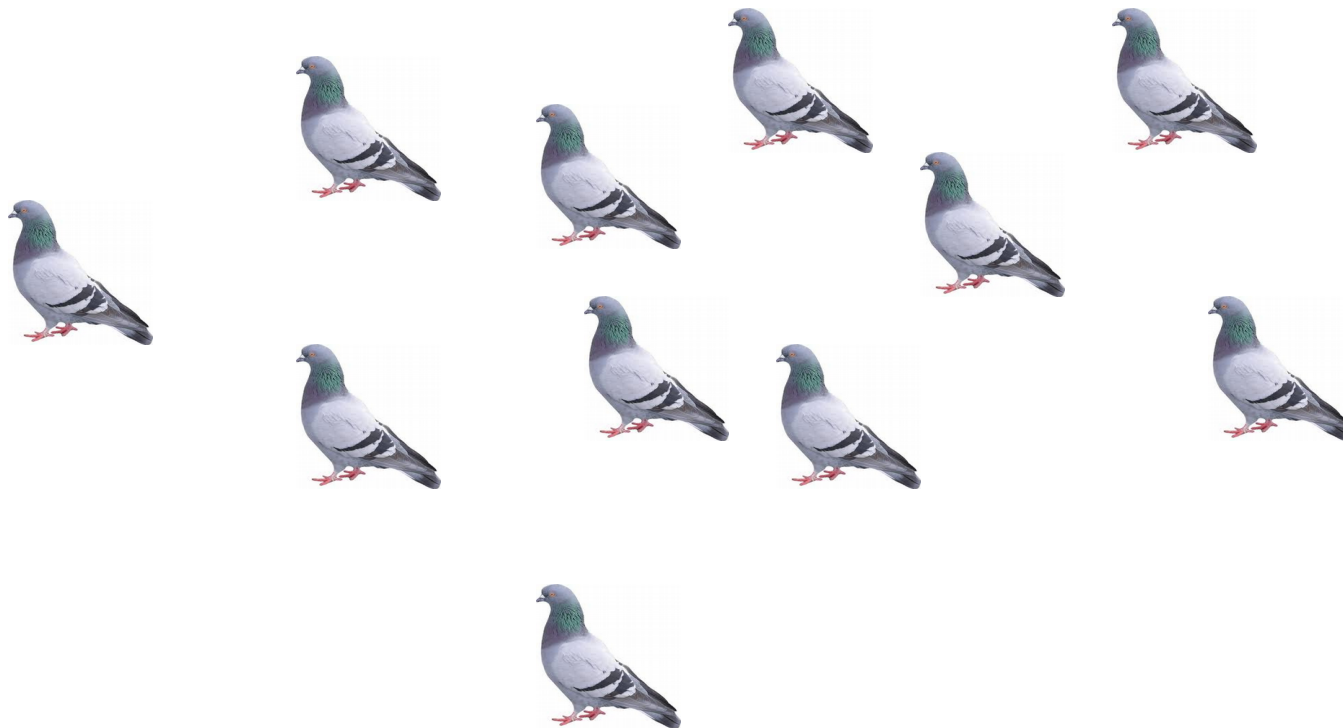
The Pigeonhole Principle



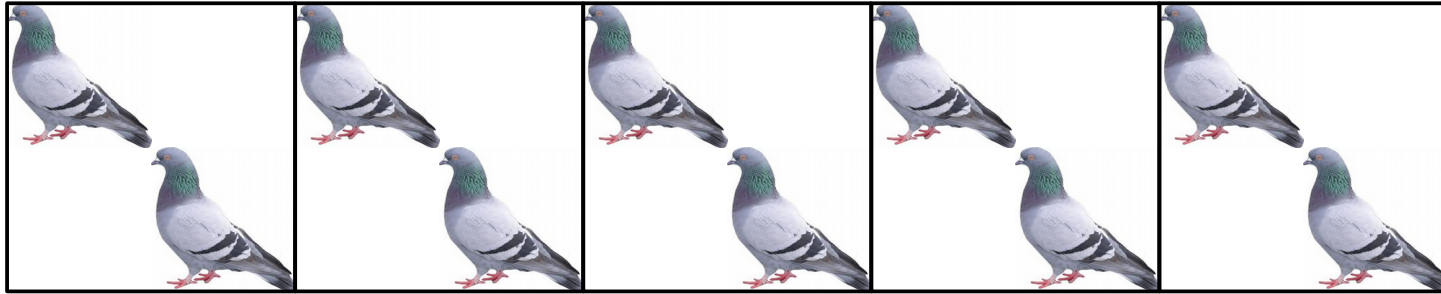
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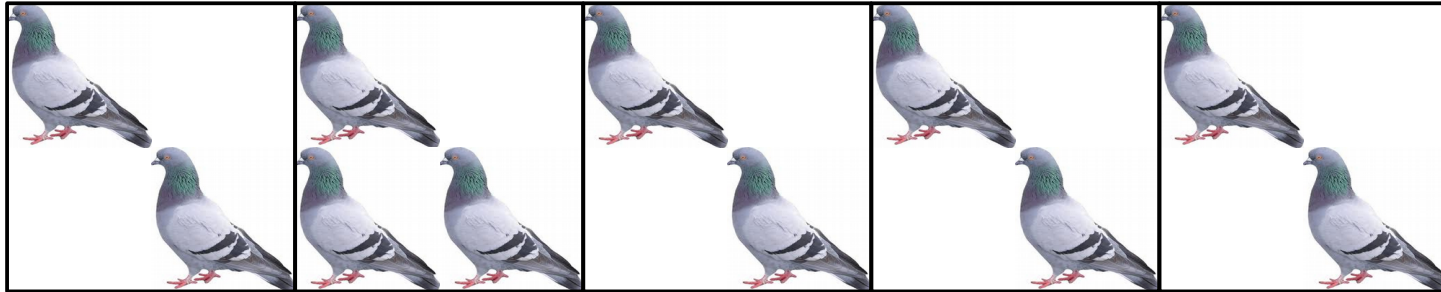
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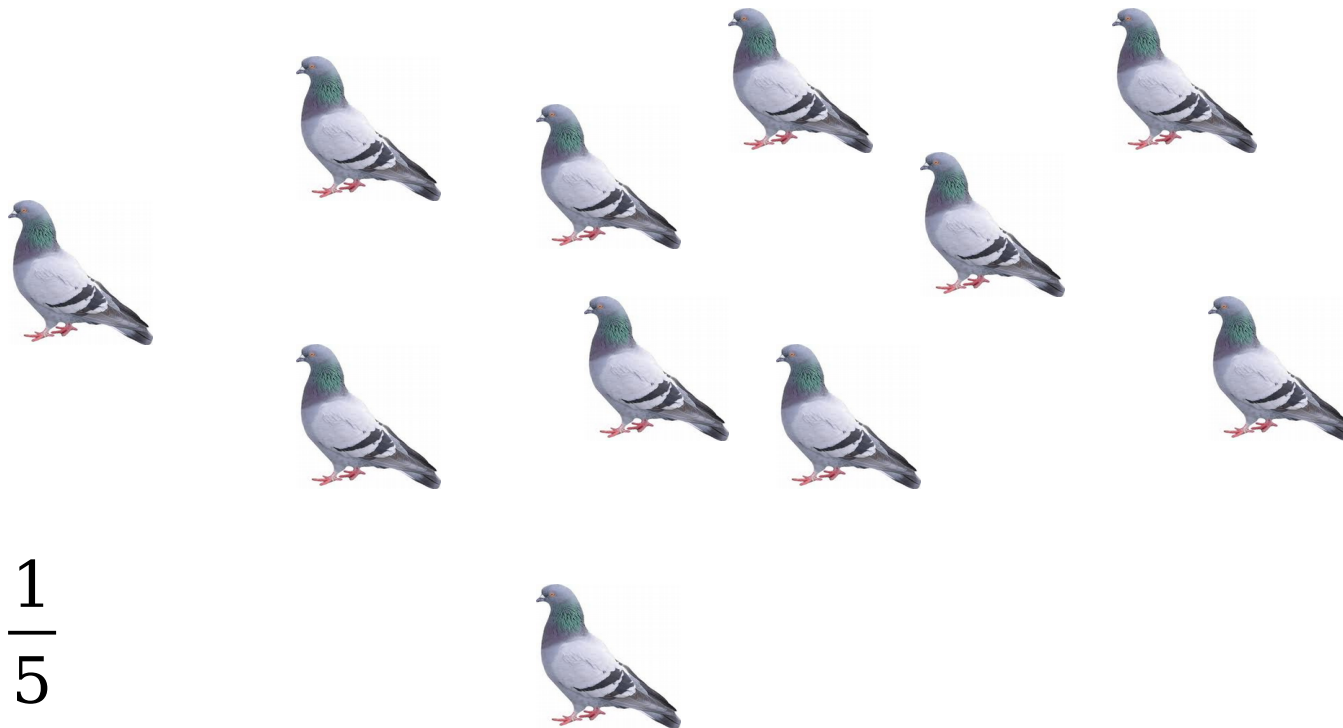
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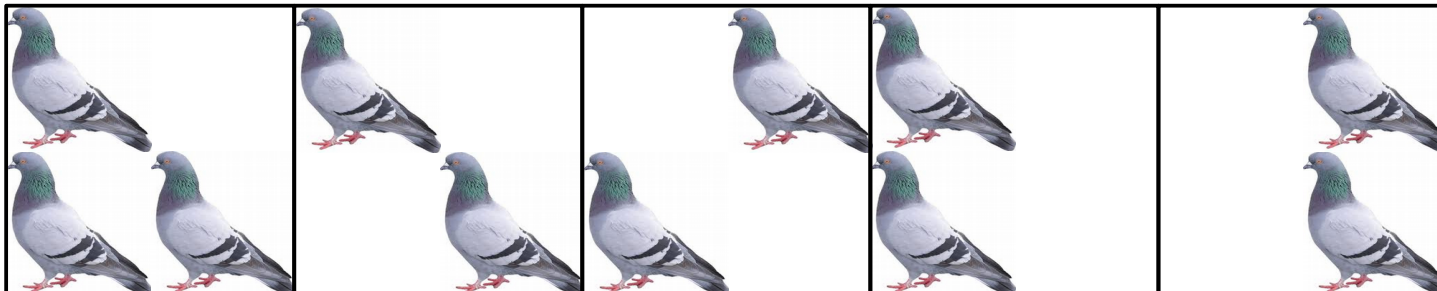


$$\frac{11}{5} = 2\frac{1}{5}$$

A More General Version

- The **generalized pigeonhole principle** says that if you distribute m objects into n bins, then
 - some bin will have at least $\lceil m/n \rceil$ objects in it, and
 - some bin will have at most $\lfloor m/n \rfloor$ objects in it.

$\lceil m/n \rceil$ means “ m/n , rounded up.”
 $\lfloor m/n \rfloor$ means “ m/n , rounded down.”



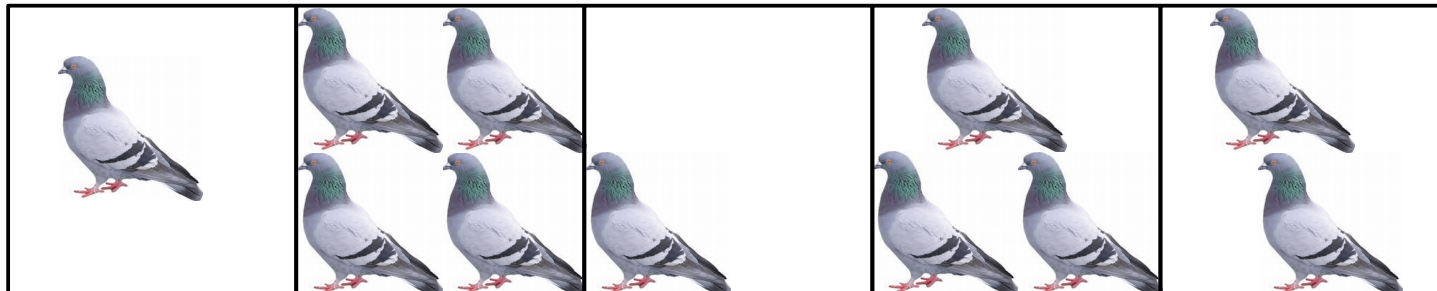
$$m = 11$$
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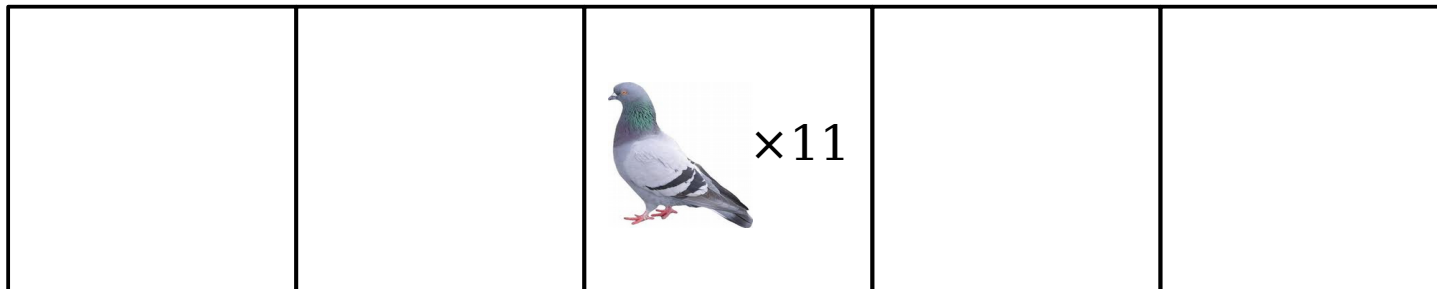
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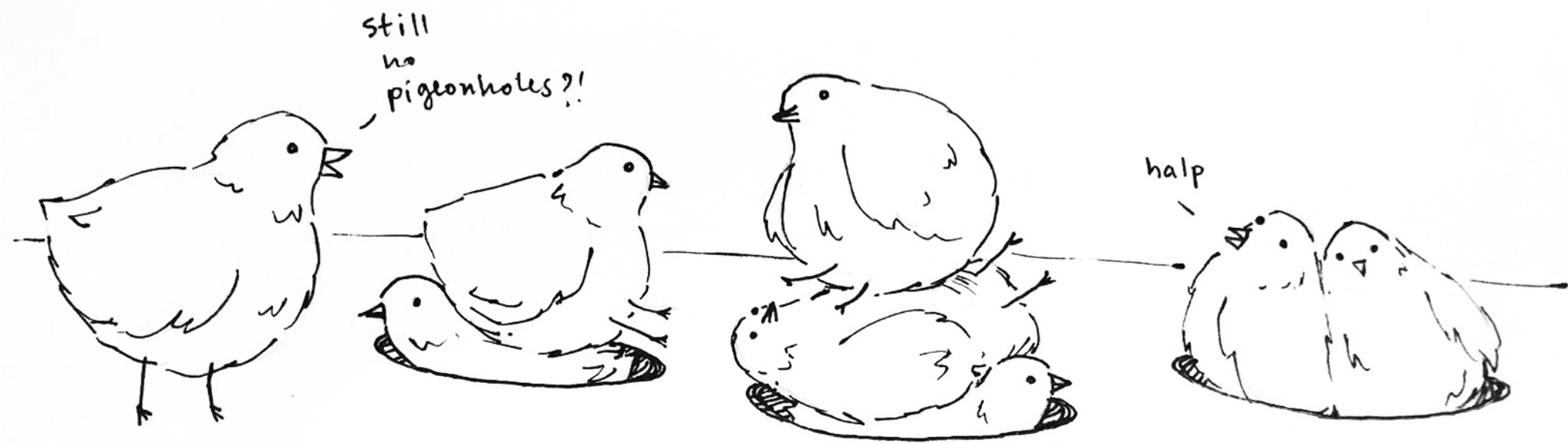
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$$m = 11$$
$$n = 5$$

$$\lceil m / n \rceil = 3$$
$$\lfloor m / n \rfloor = 2$$



$$m = 8, n = 3$$

Theorem: If m objects are distributed into $n > 0$ bins, then some bin will contain at least $\lceil m/n \rceil$ objects.

Proof: We will prove that if m objects are distributed into n bins, then some bin contains at least m/n objects. Since the number of objects in each bin is an integer, this will prove that some bin must contain at least $\lceil m/n \rceil$ objects.

To do this, we proceed by contradiction. Suppose that, for some m and n , there is a way to distribute m objects into n bins such that each bin contains fewer than m/n objects.

Number the bins $1, 2, 3, \dots, n$ and let x_i denote the number of objects in bin i . Since there are m objects in total, we know that

$$m = x_1 + x_2 + \dots + x_n.$$

Since each bin contains fewer than m/n objects, we see that $x_i < m/n$ for each i . Therefore, we have that

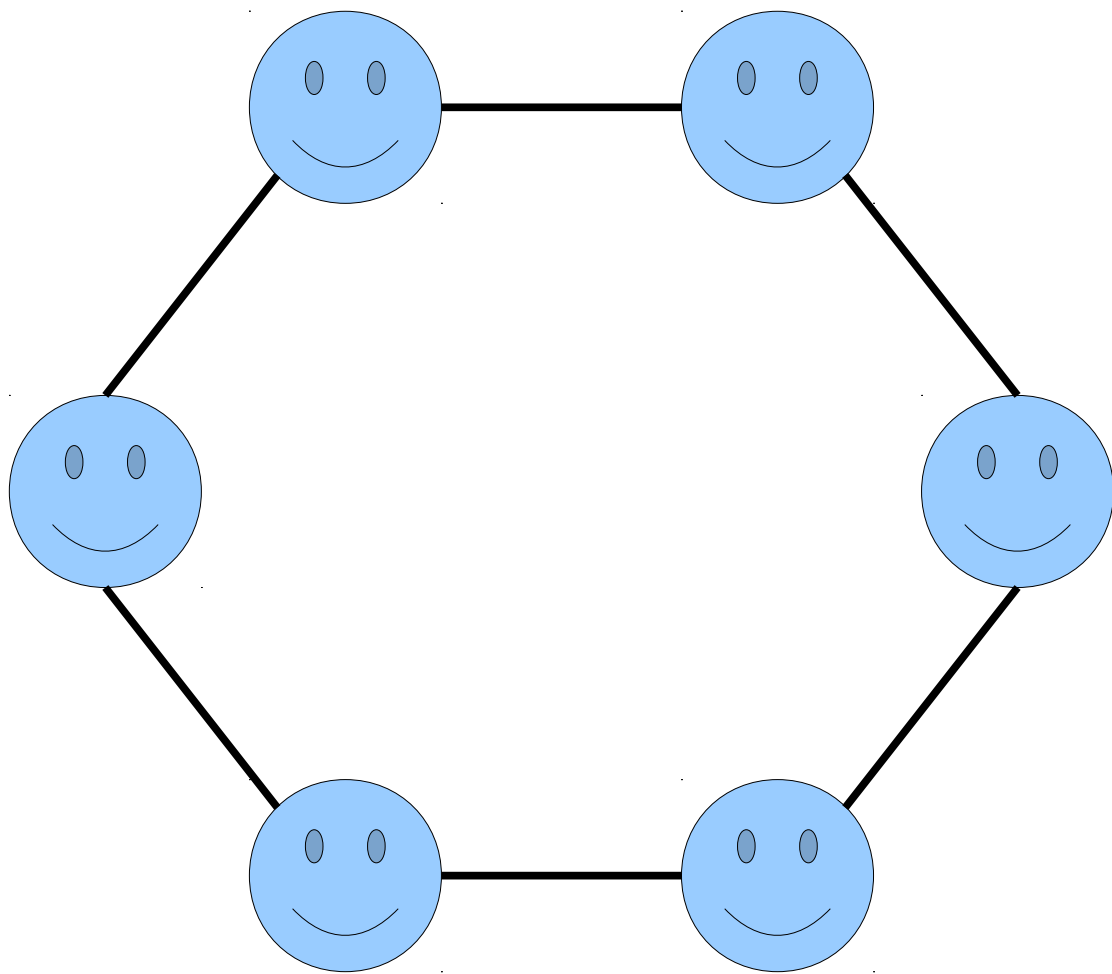
$$\begin{aligned} m &= x_1 + x_2 + \dots + x_n \\ &< m/n + m/n + \dots + m/n \quad (n \text{ times}) \\ &= m. \end{aligned}$$

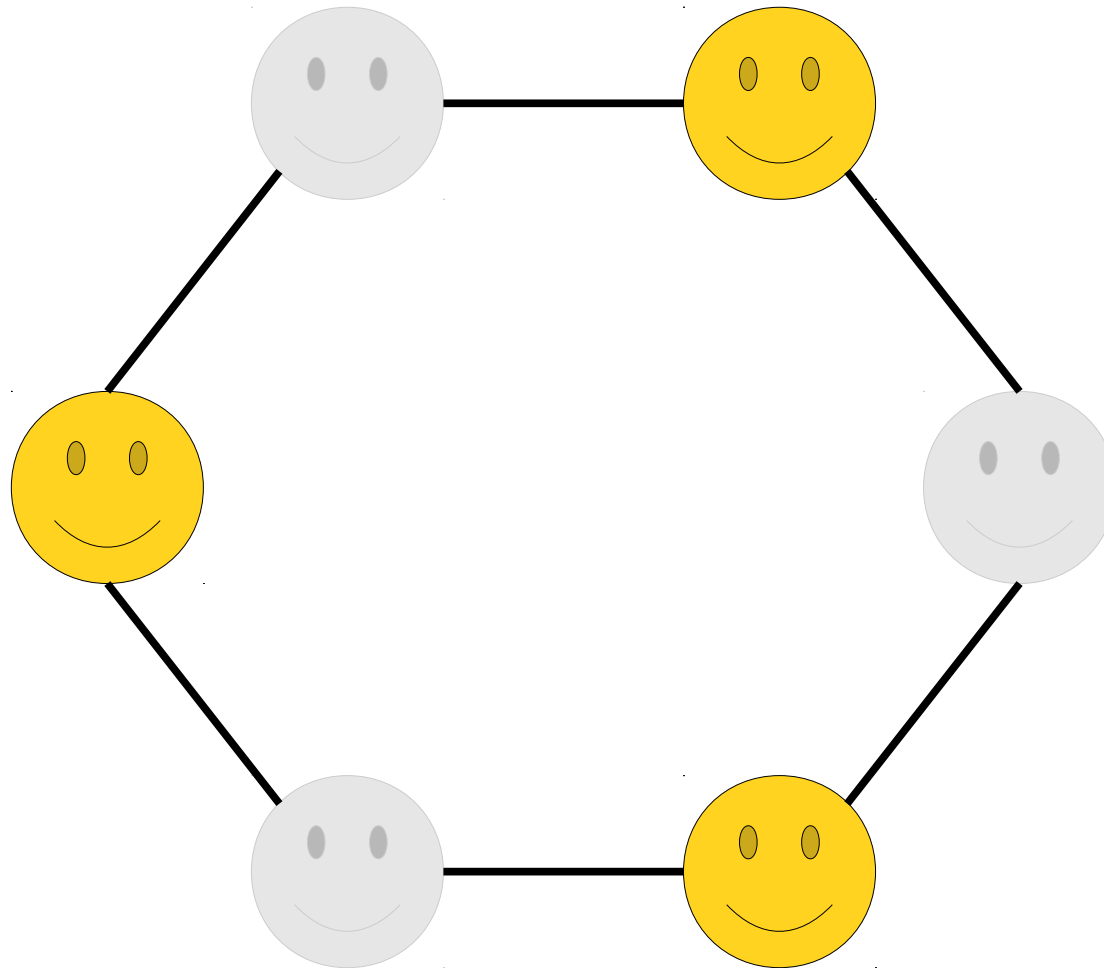
But this means that $m < m$, which is impossible. We have reached a contradiction, so our initial assumption must have been wrong. Therefore, if m objects are distributed into n bins, some bin must contain at least $\lceil m/n \rceil$ objects. ■

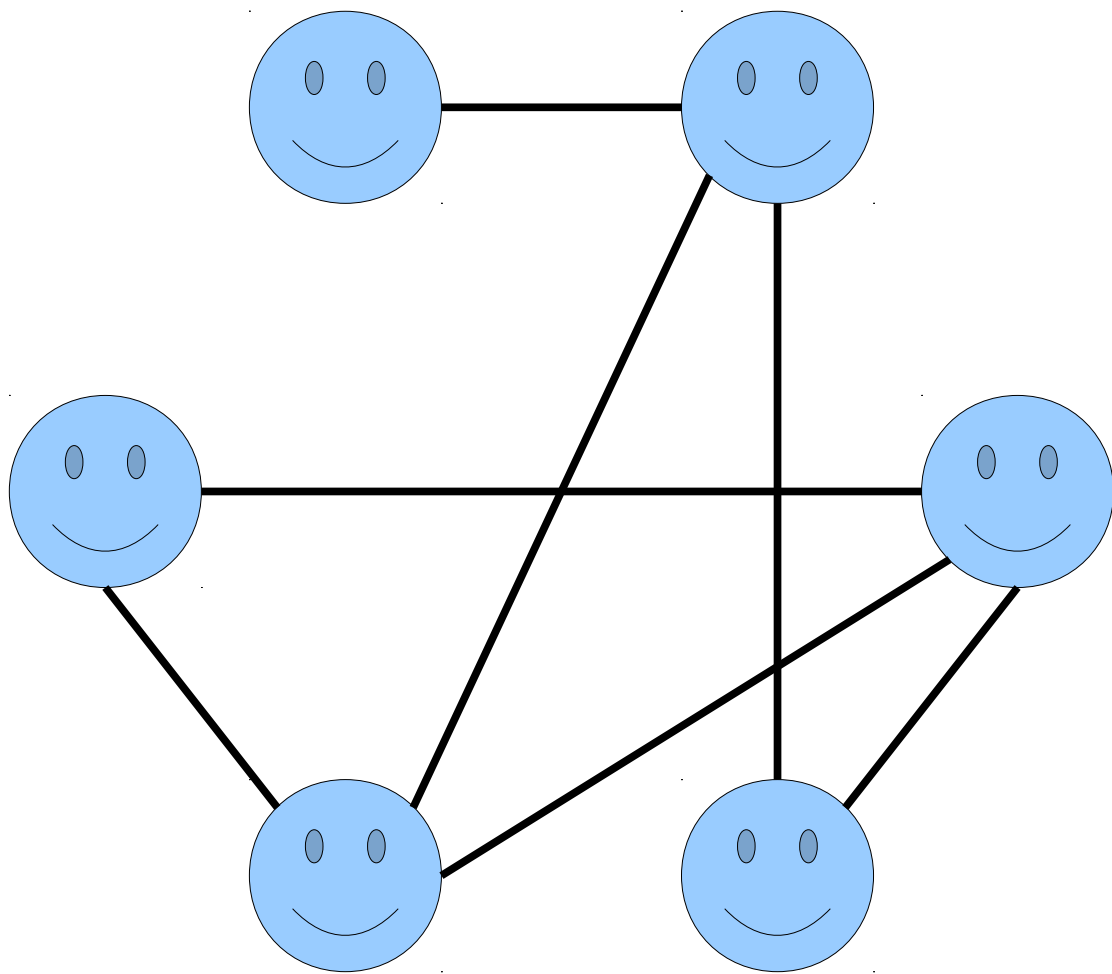
An Application: Friends and Strangers

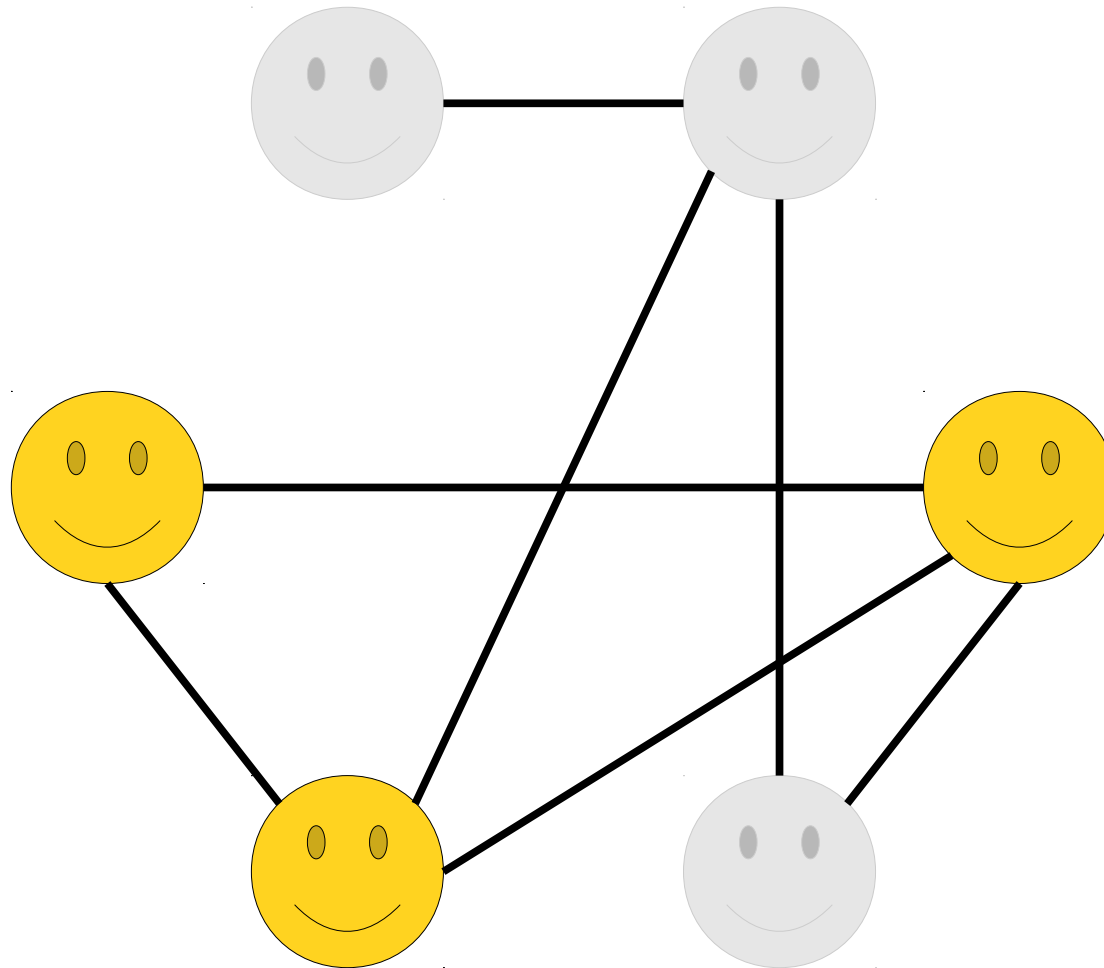
Friends and Strangers

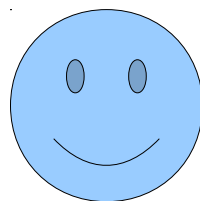
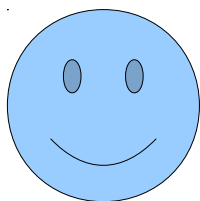
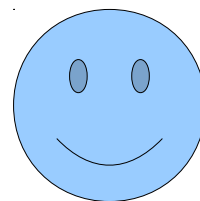
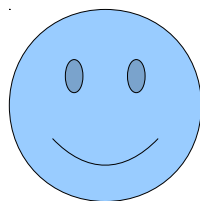
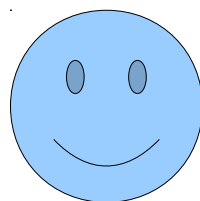
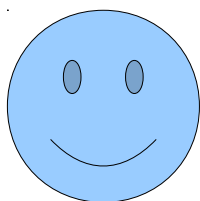
- Suppose you have a party of six people. Each pair of people are either friends (they know each other) or strangers (they do not).
- ***Theorem:*** Any such party must have a group of three mutual friends (three people who all know one another) or three mutual strangers (three people, none of whom know any of the others).

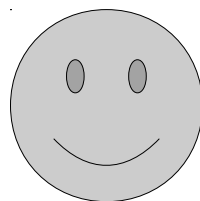
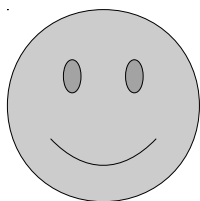
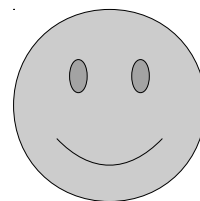
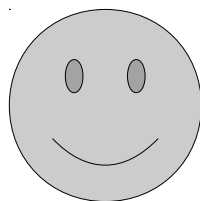
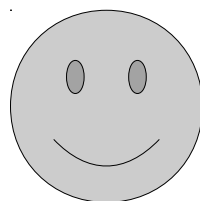
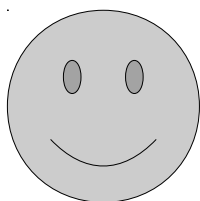


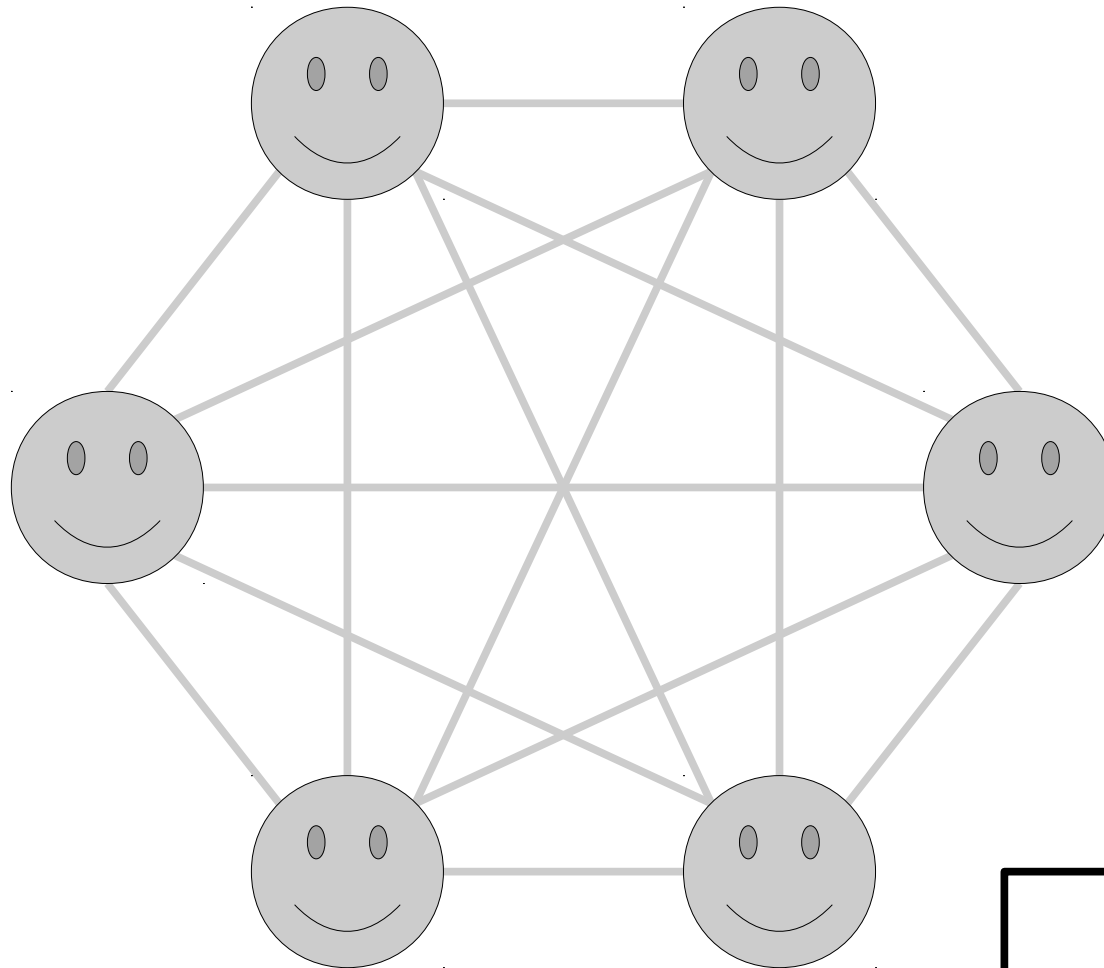




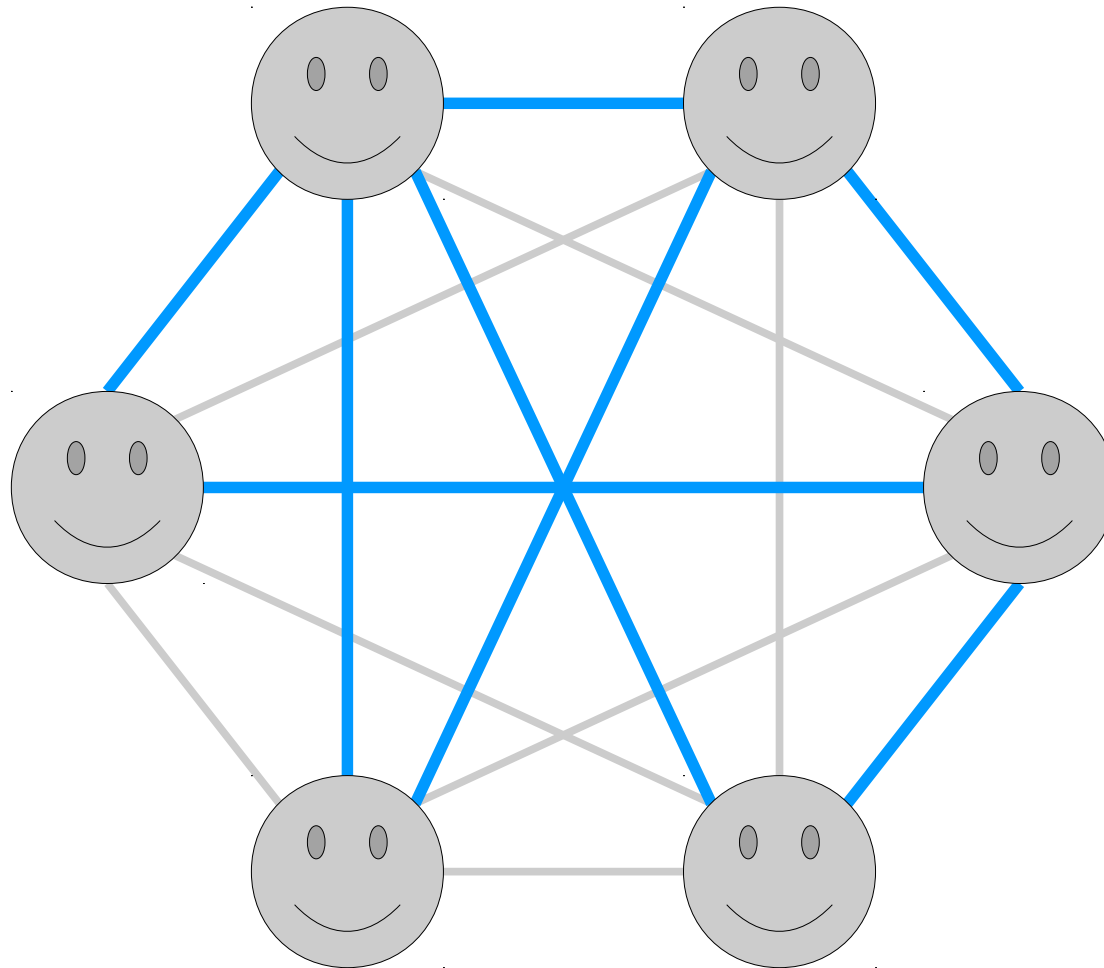


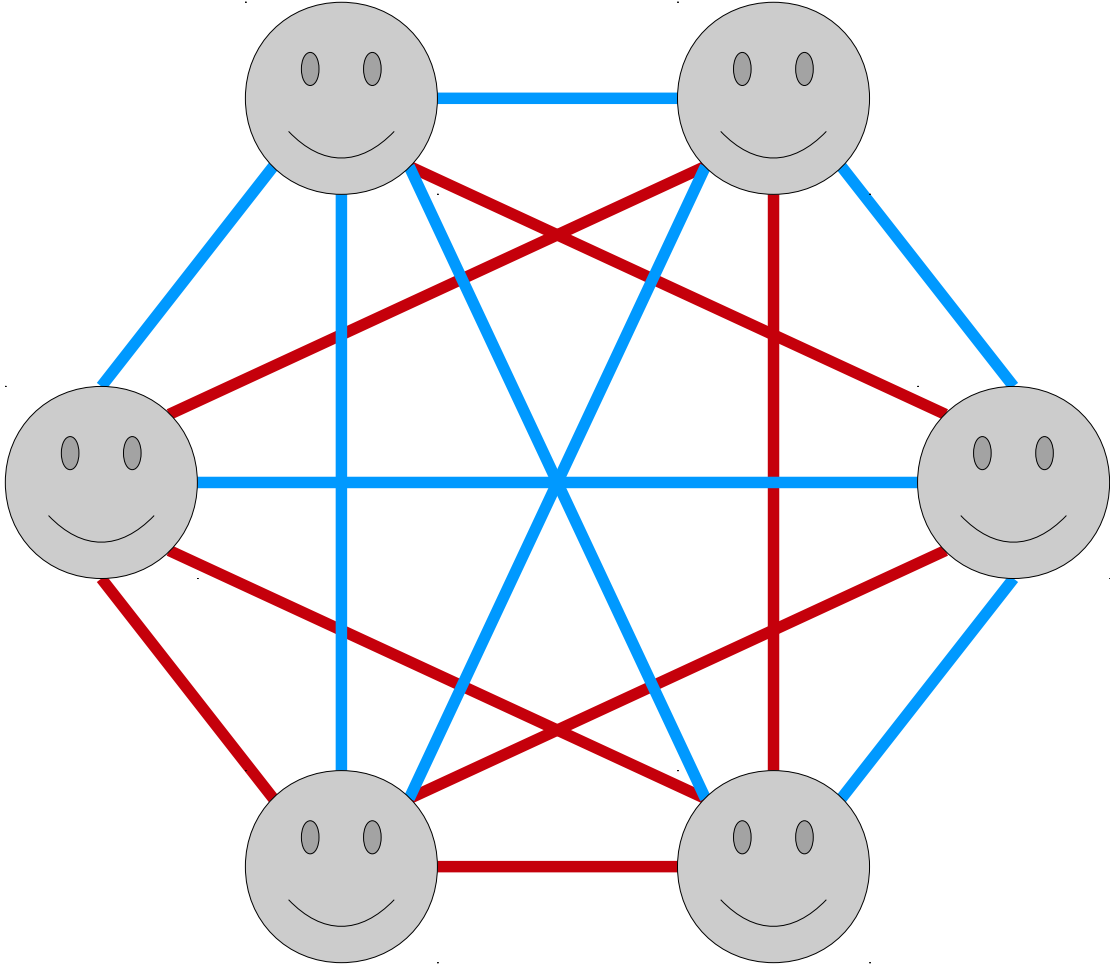


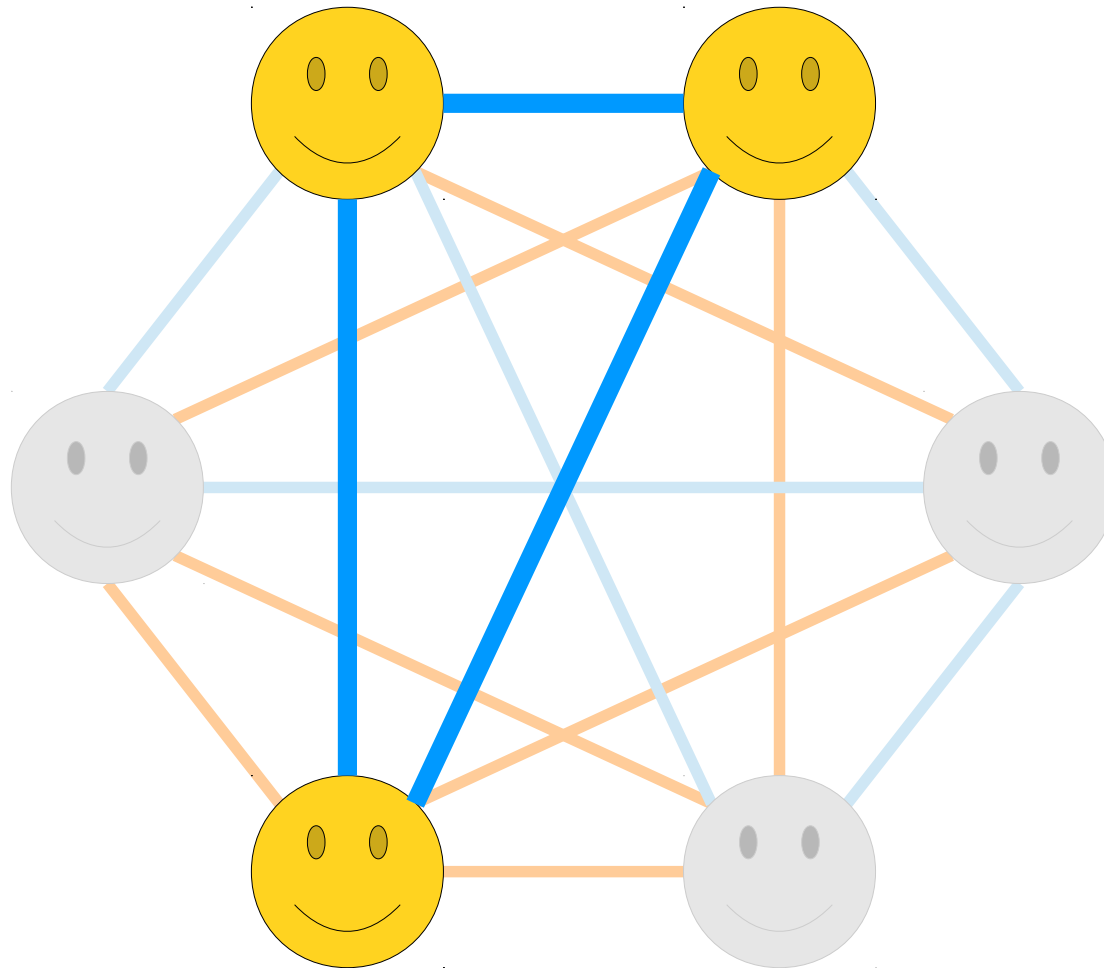


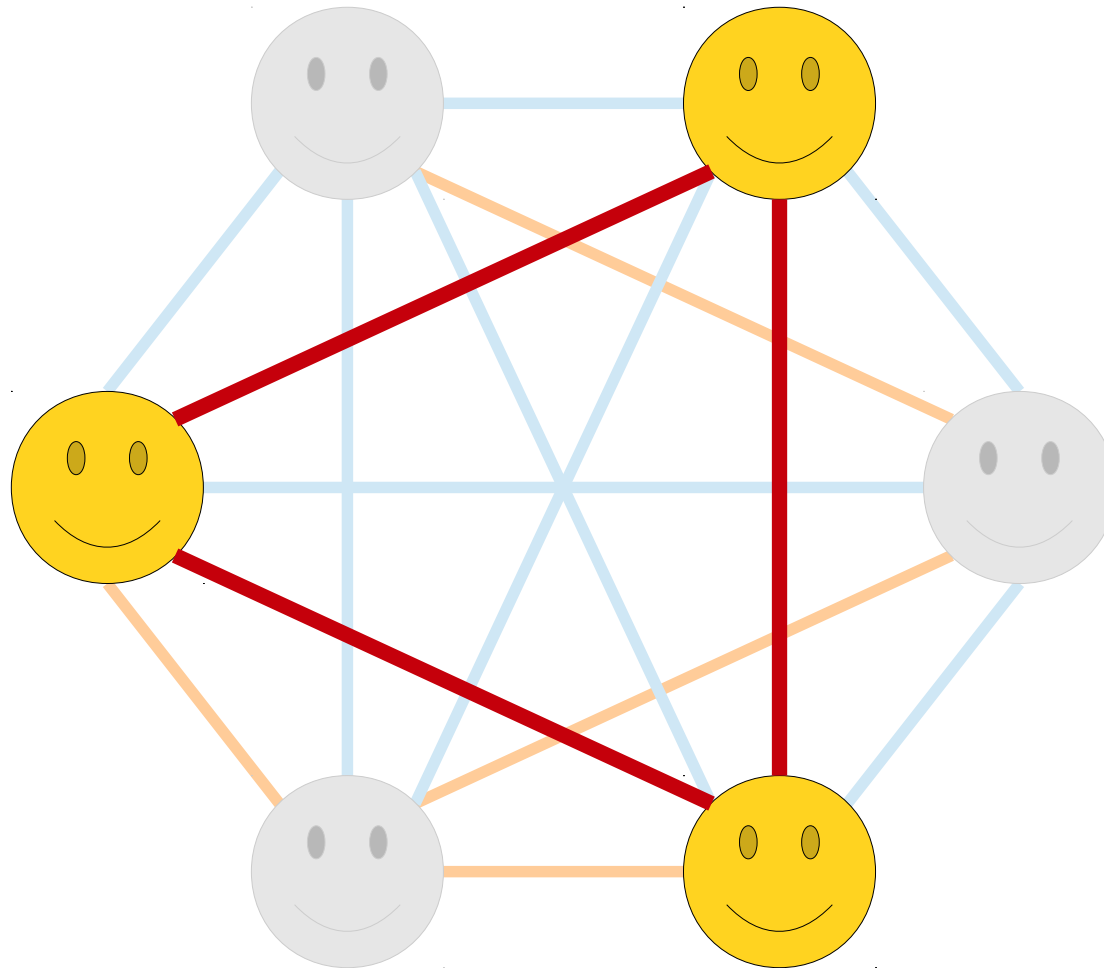


This graph is called a *6-clique*, by the way.







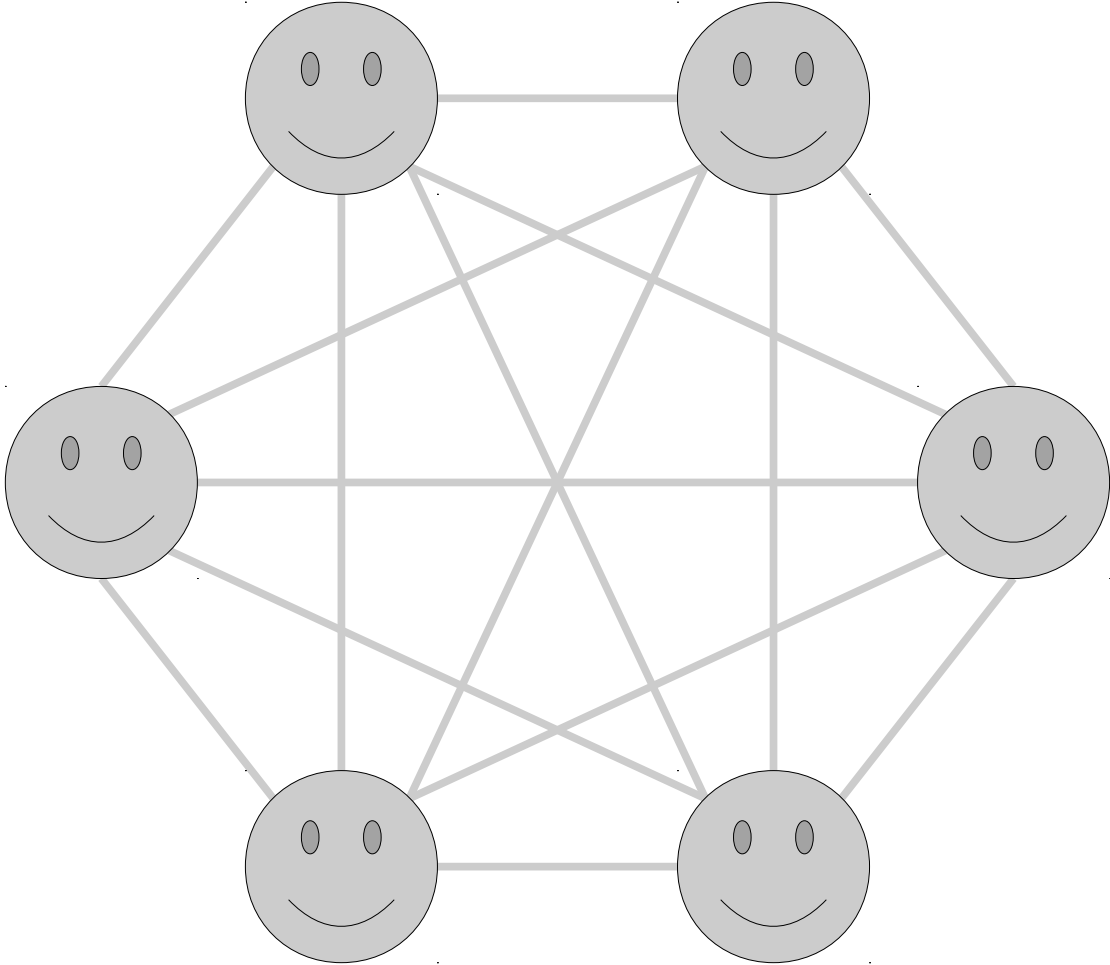


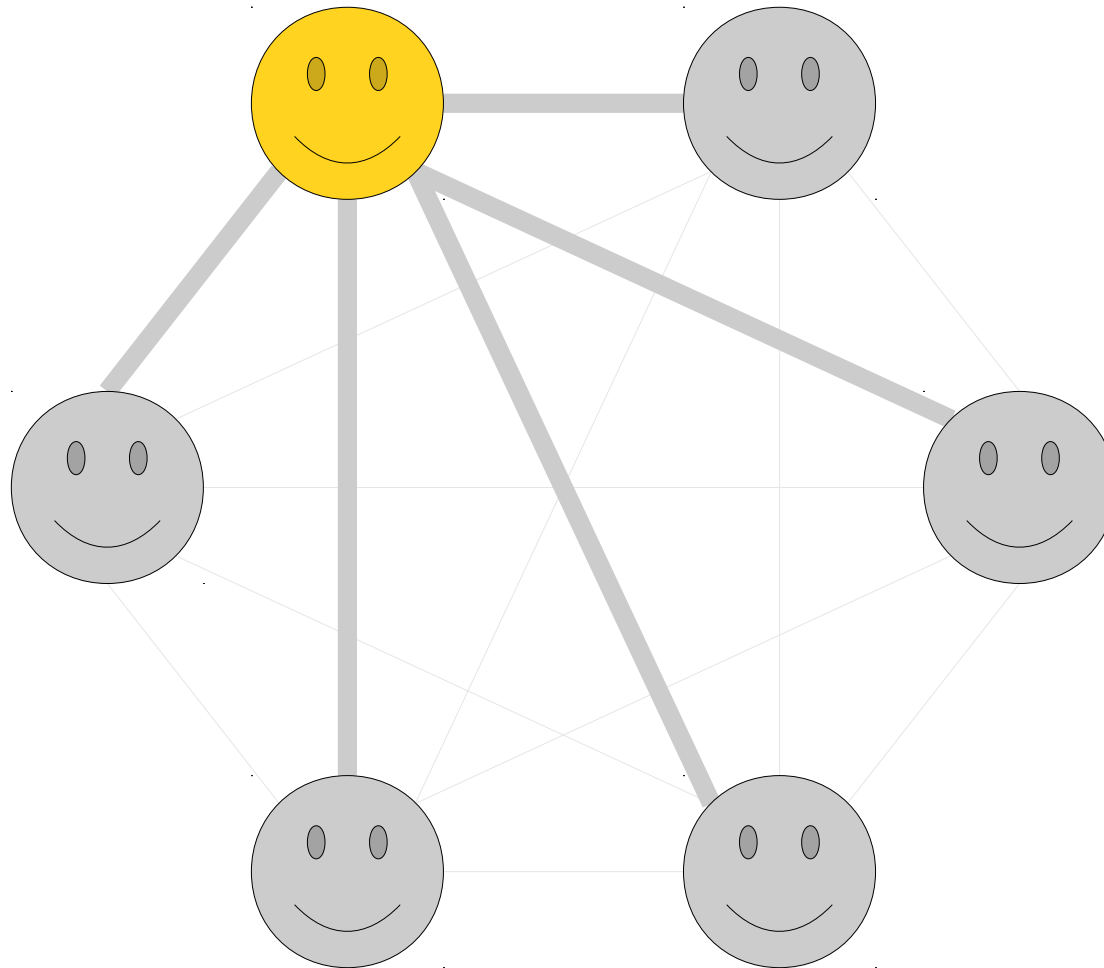
Friends and Strangers Restated

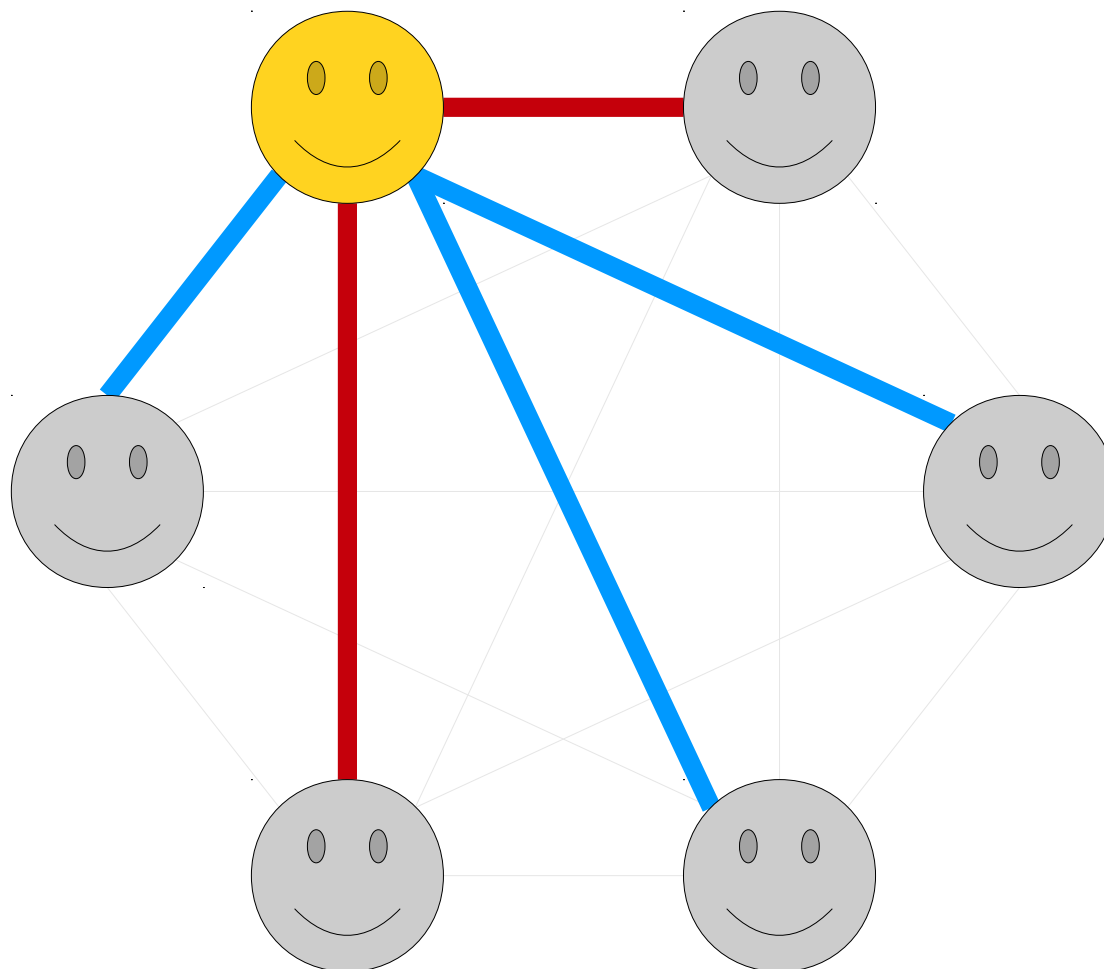
- From a graph-theoretic perspective, the Theorem on Friends and Strangers can be restated as follows:

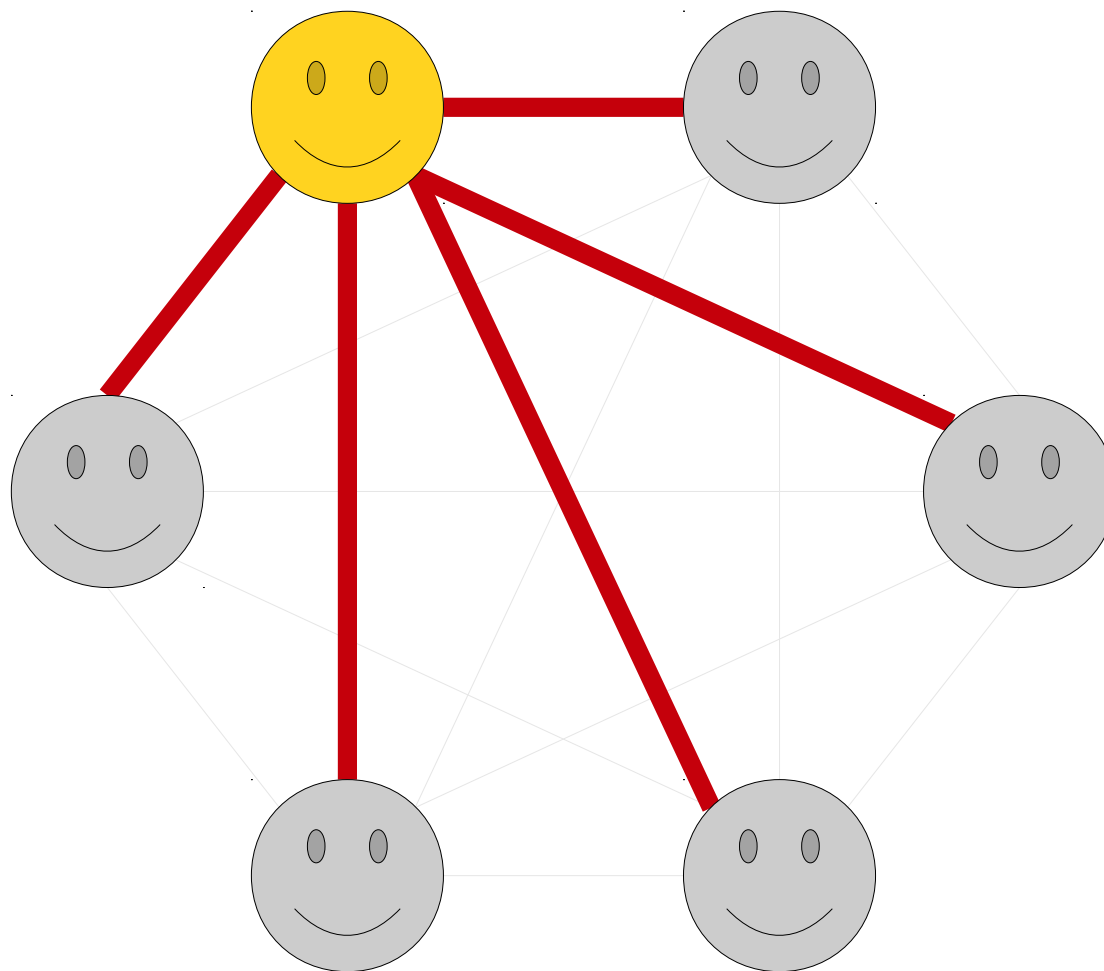
Theorem: Any 6-clique whose edges are colored red and blue contains a red triangle or a blue triangle (or both).

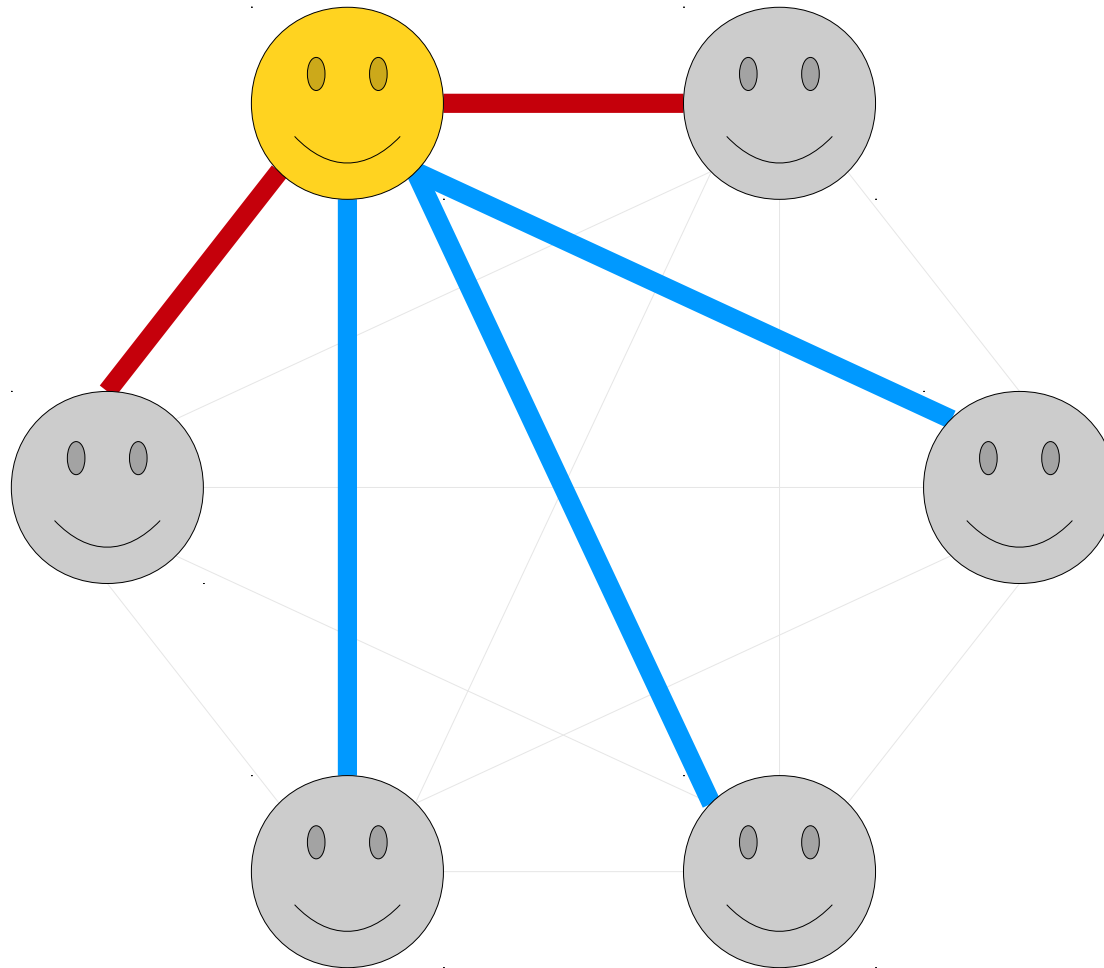
- How can we prove this?

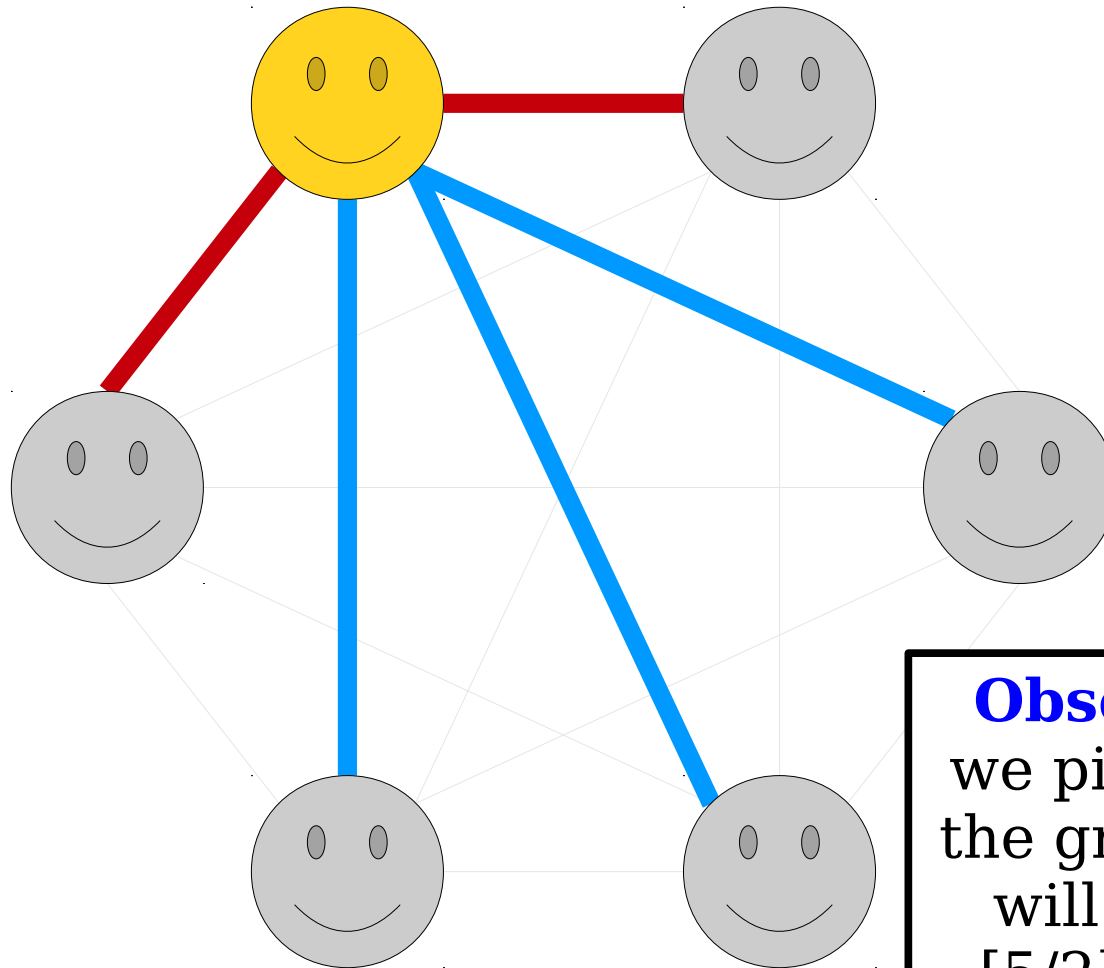




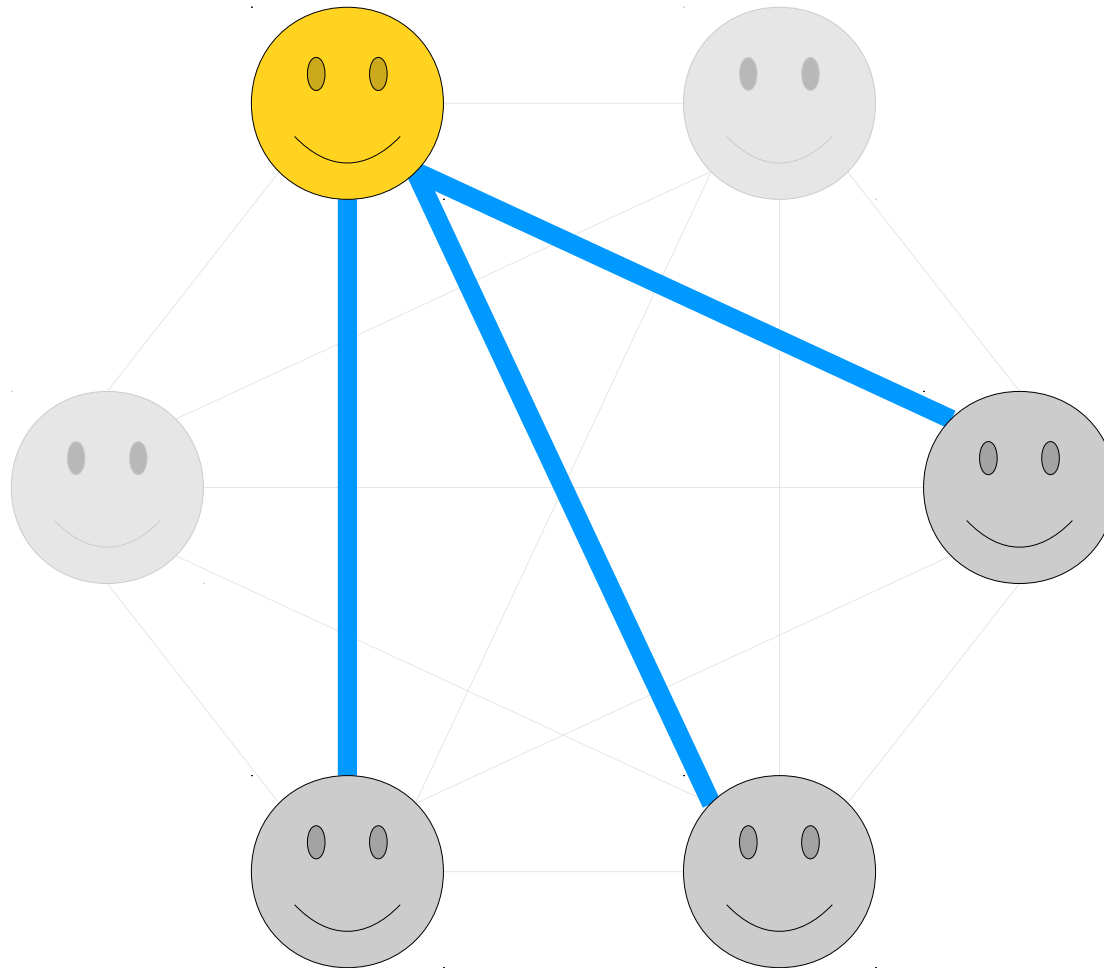


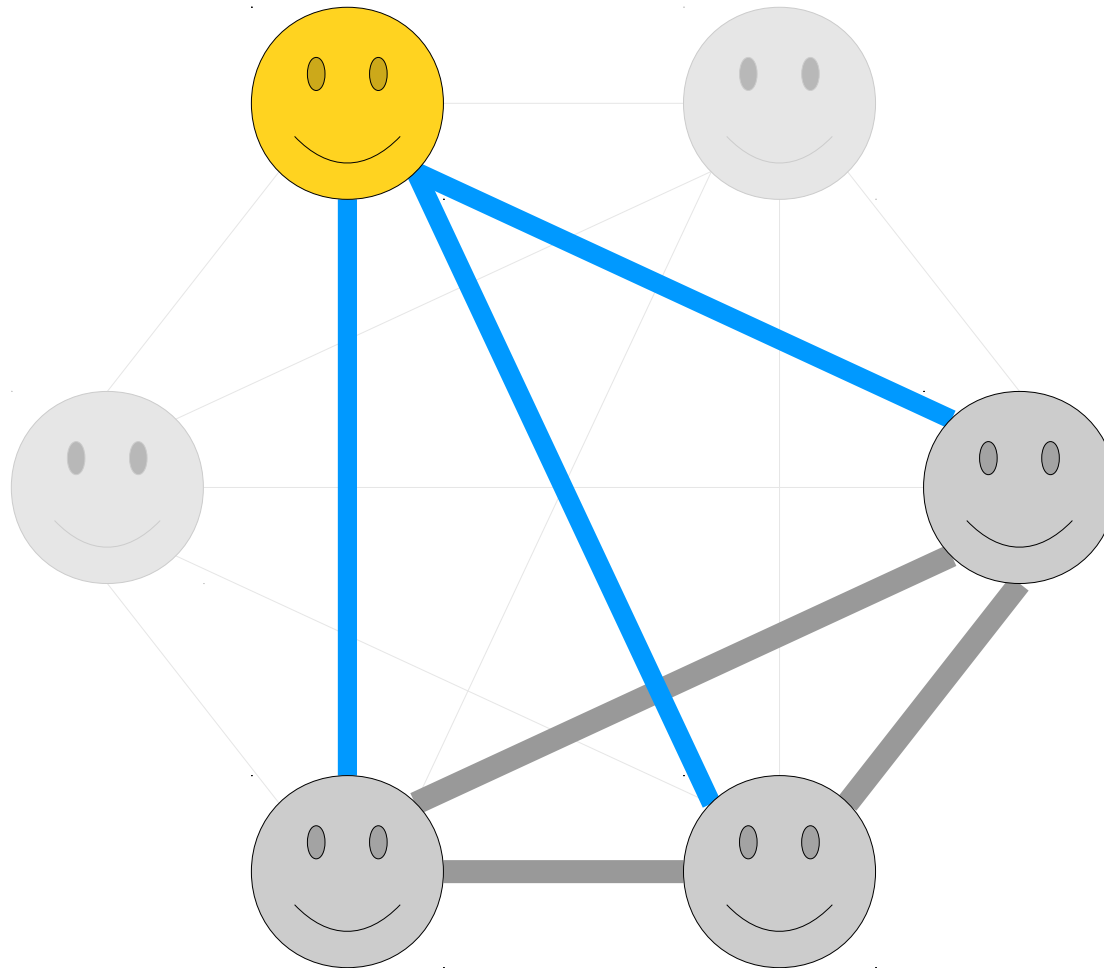


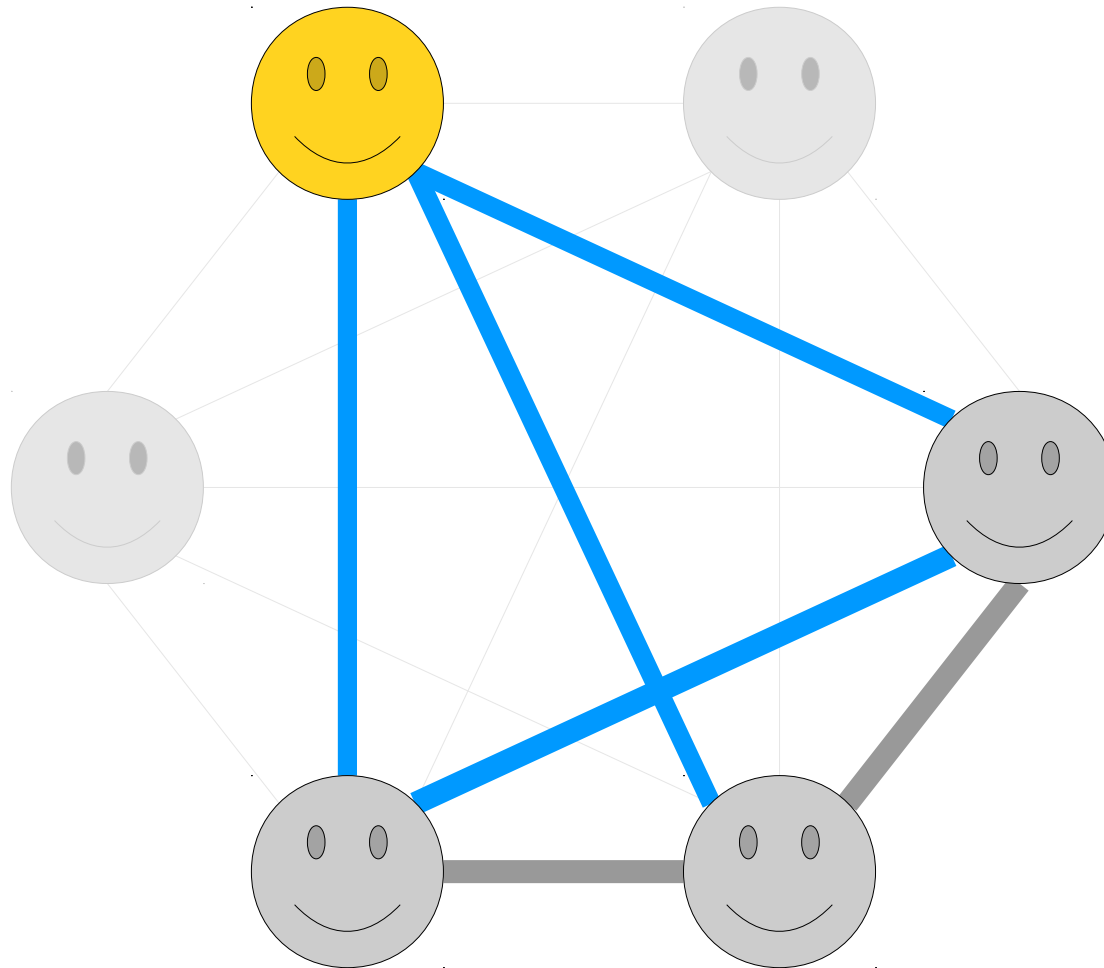


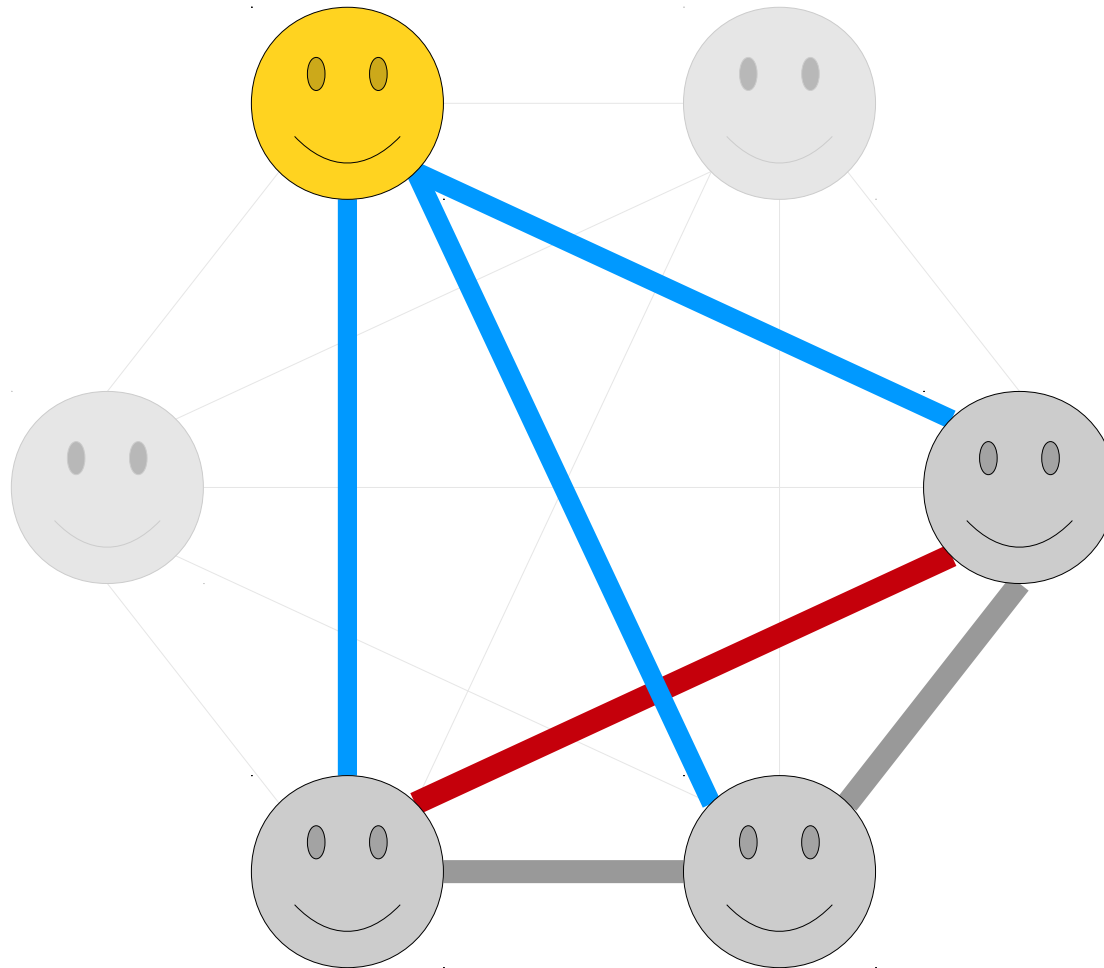


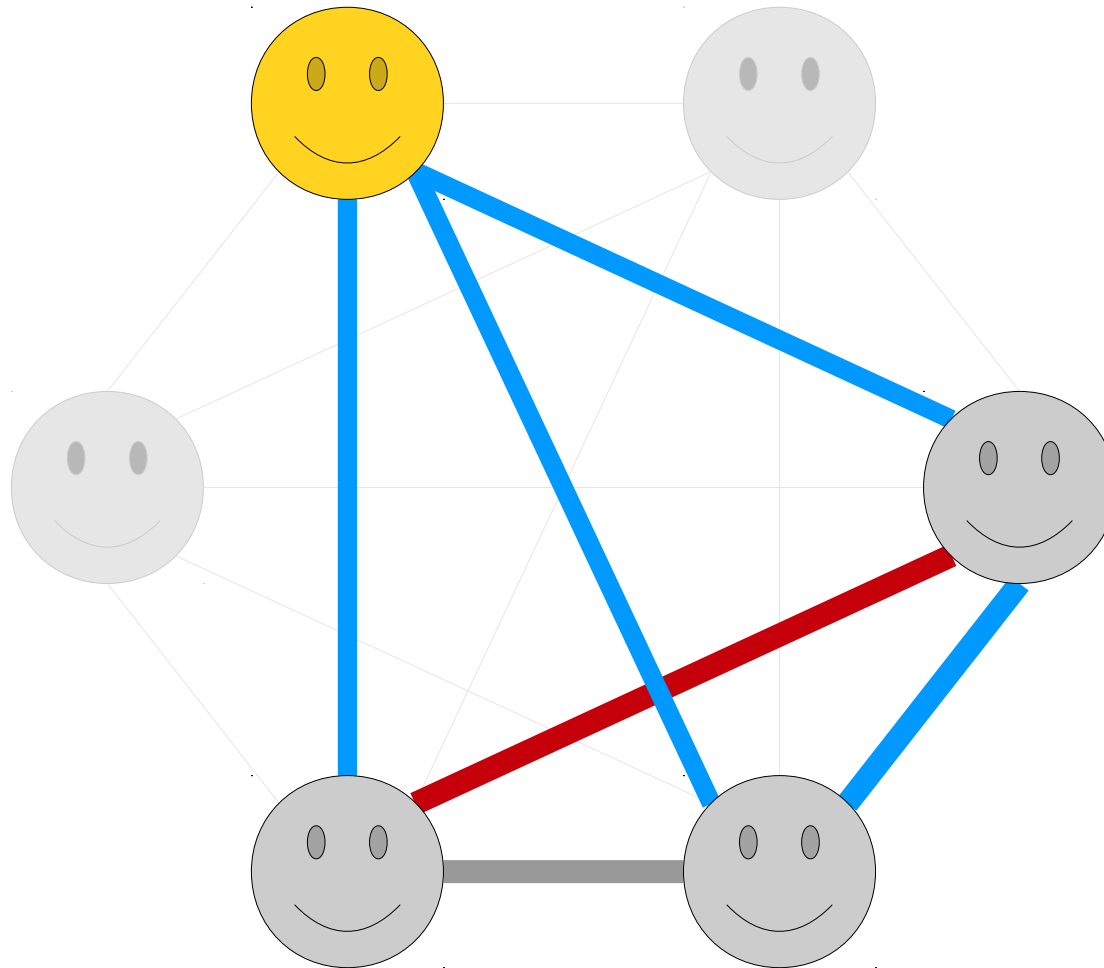
Observation 1: If we pick any node in the graph, that node will have at least $\lceil 5/2 \rceil = 3$ edges of the same color incident to it.

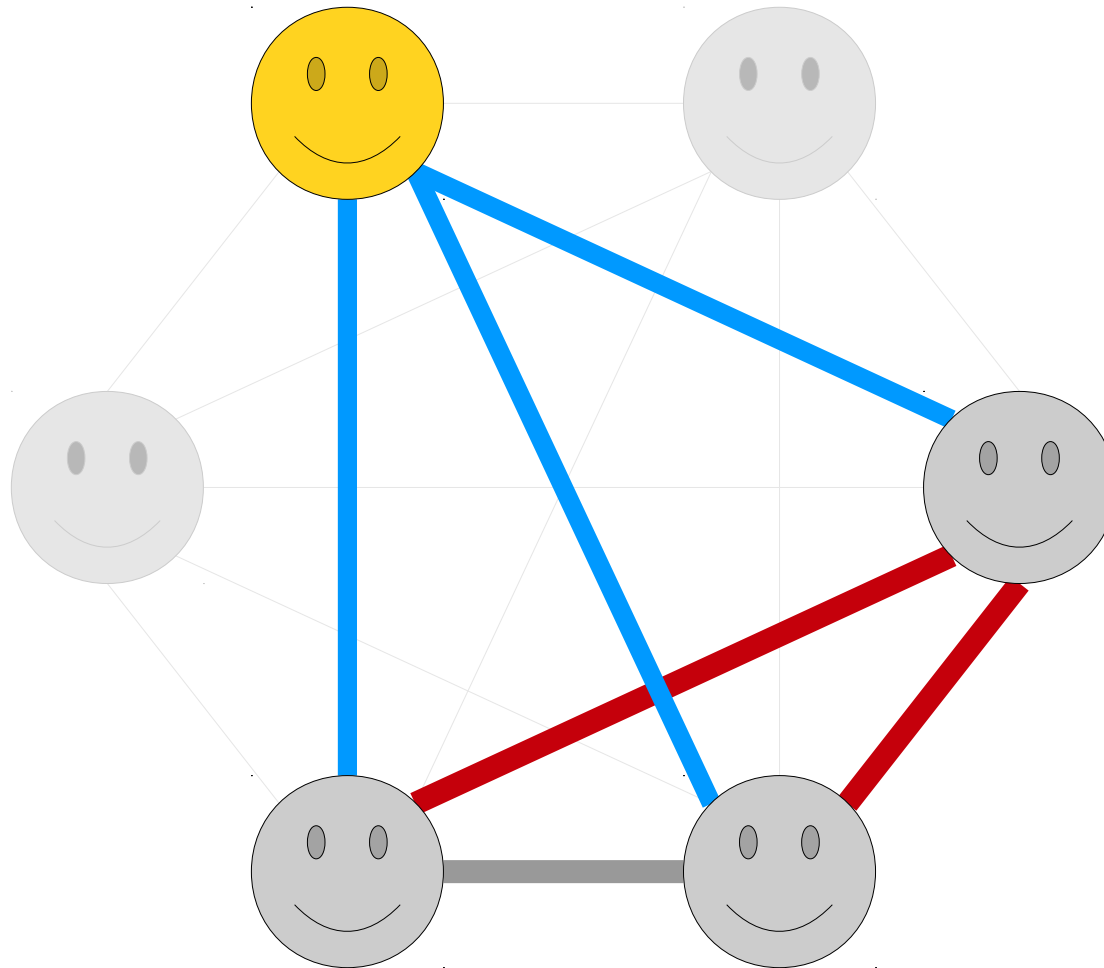


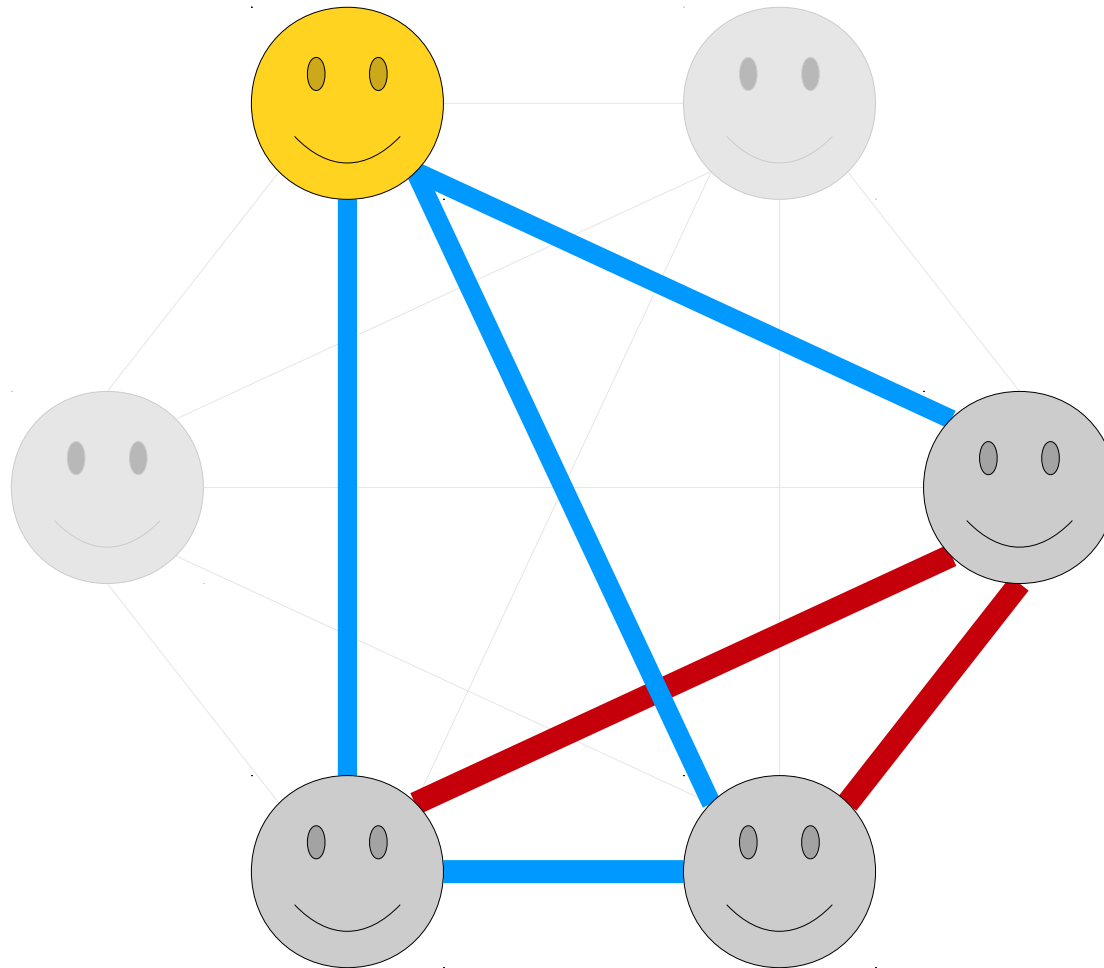


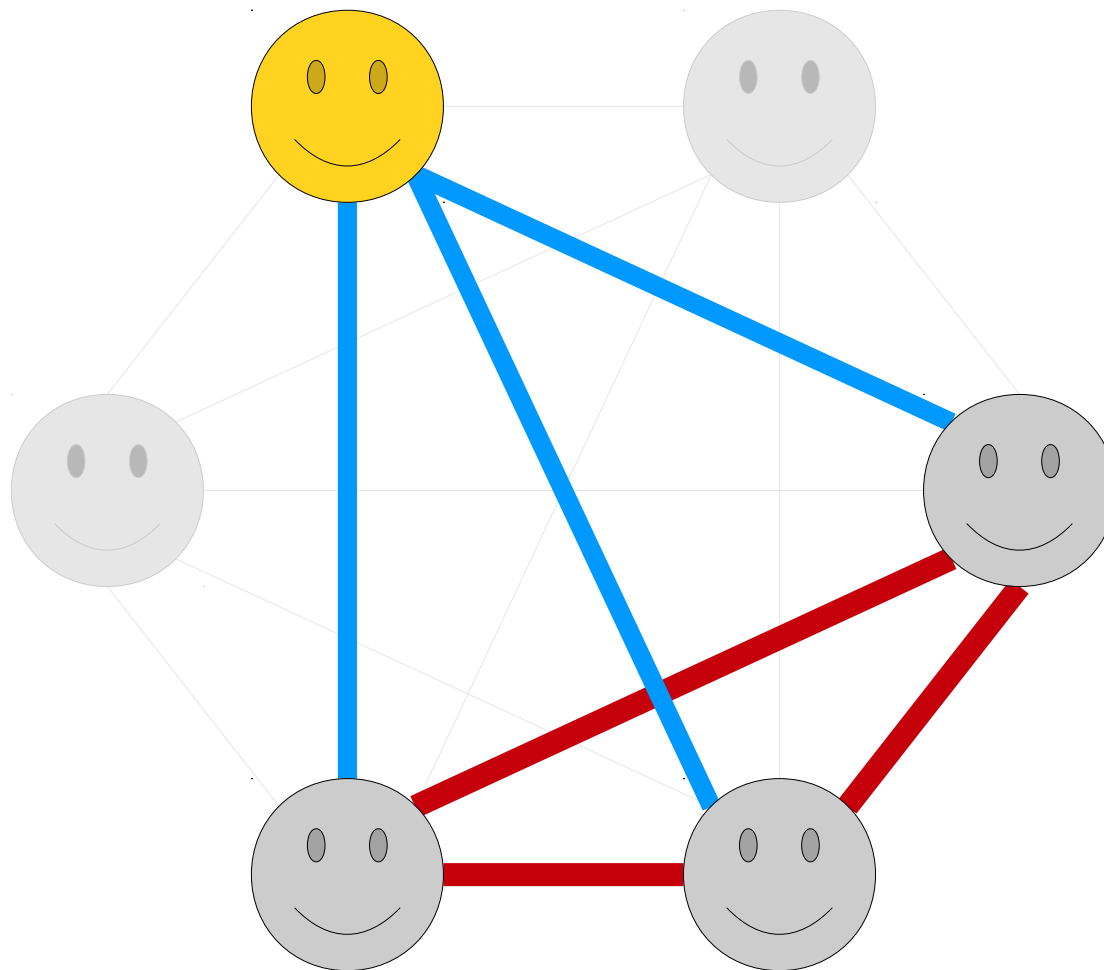












Theorem: Consider a 6-clique in which every edge is colored either red or blue. Then there must be a triangle of red edges, a triangle of blue edges, or both.

Proof: We need to show that the colored 6-clique contains a red triangle or a blue triangle.

Let x be any node in the 6-clique. It is incident to five edges and there are two possible colors for those edges. Therefore, by the generalized pigeonhole principle, at least $\lceil 5/2 \rceil = 3$ of those edges must be the same color. Without loss of generality, assume those edges are blue.

Let r , s , and t be three of the nodes adjacent to node x along a blue edge. If any of the edges $\{r, s\}$, $\{r, t\}$, or $\{s, t\}$ are blue, then one of those edges plus the two edges connecting back to node x form a blue triangle. Otherwise, all three of those edges are red, and they form a red triangle. Overall, this gives a red triangle or a blue triangle, as required. ■

Ramsey Theory

- This proof is a special case of a broader family of results called ***Ramsey theory***.
- ***Theorem (Ramsey)***: For any natural number s , there is a number $R(s)$ such that
 - for all $n < R(s)$, there's a way to color the edges of K_n red and blue so there are no monochrome copies of K_s , and
 - for all $n \geq R(s)$, every way of coloring the edges of K_n red and blue always has a monochrome copy of K_s .
- Take Math 108 (combinatorics) to learn more!
- A more philosophical (and less literal) take on this theorem: true disorder is impossible at a large scale, since no matter how you organize things, you're guaranteed to find some interesting substructure.

Time-Out for Announcements!

Problem Sets

- Problem Set Three was due today at 4:00PM.
 - You can use a late day to extend the deadline to Saturday at 4:00PM if you'd like.
- Problem Set Four goes out today. It's due next Friday at 4:00PM.
 - It's all about graphs and graph theory, and you'll see some really cool results!
 - Because the midterm is on Tuesday, we've made this problem set shorter than the previous problem sets.

Midterm Logistics

- Our first midterm is next Tuesday from 7PM – 10PM. Locations are divvied up by last (family) name:
 - A – P: Go to Hewlett 200.
 - Q – Z: Go to Hewlett 201.
- Best of luck on the exam – ***you can do this!*** We're all cheering you on.
- There's plenty of extra problems online if you're looking to get some additional practice.
- Feel free to ask questions on EdStem over the weekend.
- Review session on Monday!

My Advice

- **Do** block out some dedicated time to work through practice problems.
- **Do** get the TAs to review your answers to those problems; ask privately on EdStem.
- **Do** take some time this weekend to take a walk, smell the rosemary bushes on campus, and watch the bees buzz.
- **Don't** pull an all-nighter studying for the exam.
- **Don't** skip meals or alter your daily routine to fit in time for studying.
- **Don't** panic. You can do this!

Back to CS103!

A Little Math Puzzle

“In a group of $n > 0$ people ...

- 90% of those people enjoyed *Get Out*,
- 80% of those people enjoyed *Lady Bird*,
- 70% of those people enjoyed *Arrival*, and
- 60% of those people enjoyed *Zootopia*.

No one enjoyed all four movies. How many people enjoyed at least one of *Get Out* and *Arrival*?”

Other Pigeonhole-Type Results

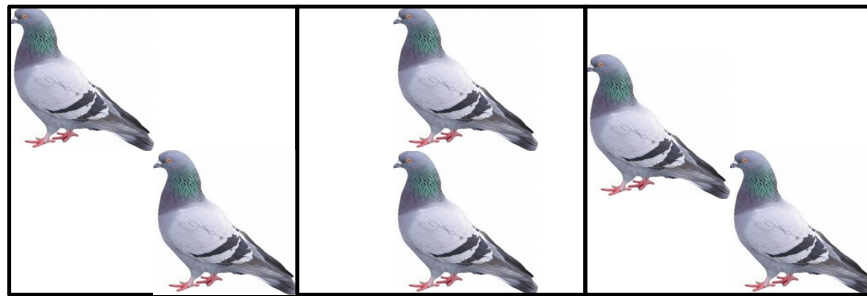
*If m objects are distributed into n boxes, then **[condition]** holds.*

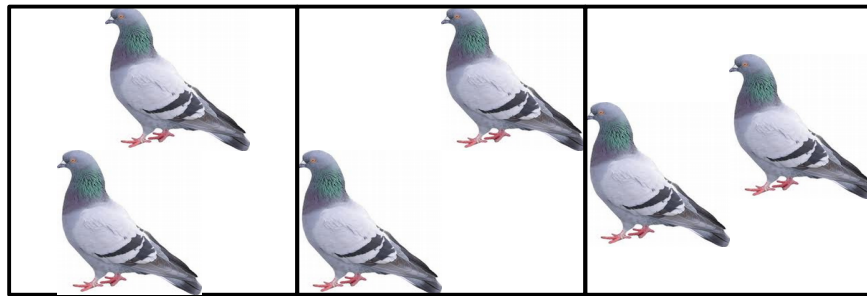
*If m objects are distributed into n boxes, then **some box is loaded to at least the average m/n , and some box is loaded to at most the average m/n .***

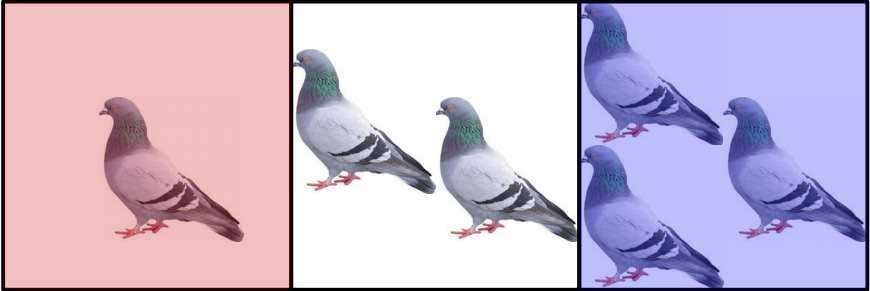
*If m objects are distributed into n boxes, then **[condition]** holds.*



--	--	--









Theorem: If m objects are distributed into n bins, then there is a bin containing more than m/n objects if and only if there is a bin containing fewer than m/n objects.

Lemma: If m objects are distributed into n bins and there are no bins containing more than m/n objects, then there are no bins containing fewer than m/n objects.

Lemma: If m objects are distributed into n bins and there are no bins containing more than m/n objects, then there are no bins containing fewer than m/n objects.

Proof: Assume for the sake of contradiction that m objects are distributed into n bins such that no bin contains more than m/n objects, yet some bin has fewer than m/n objects.

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For simplicity, denote by x_i the number of objects in bin i .

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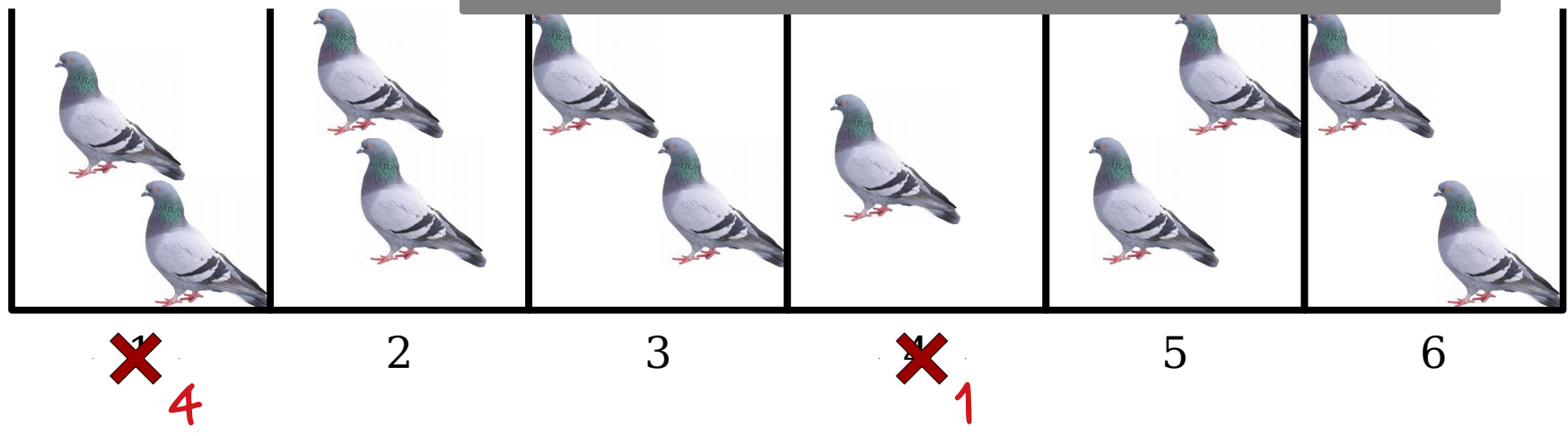
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This magic phrase means "we get to pick how we're labeling things anyway, so if it doesn't work out, just relabel things."



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For simplicity, denote by x_i the number of objects in bin i . Without loss of generality, assume that bin 1 has fewer than m/n objects, meaning that $x_1 < m/n$. Adding up the number of objects in each bin tells us that

$$m = x_1 + x_2 + x_3 + \dots + x_n$$

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This third step follows because each remaining bin has at most m/n objects.

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This third step follows because each remaining bin has at most m/n objects. Grouping the n copies of the m/n term here tells us that

$$\begin{aligned} m &< m/n + m/n + m/n + \dots + m/n \\ &= m. \end{aligned}$$

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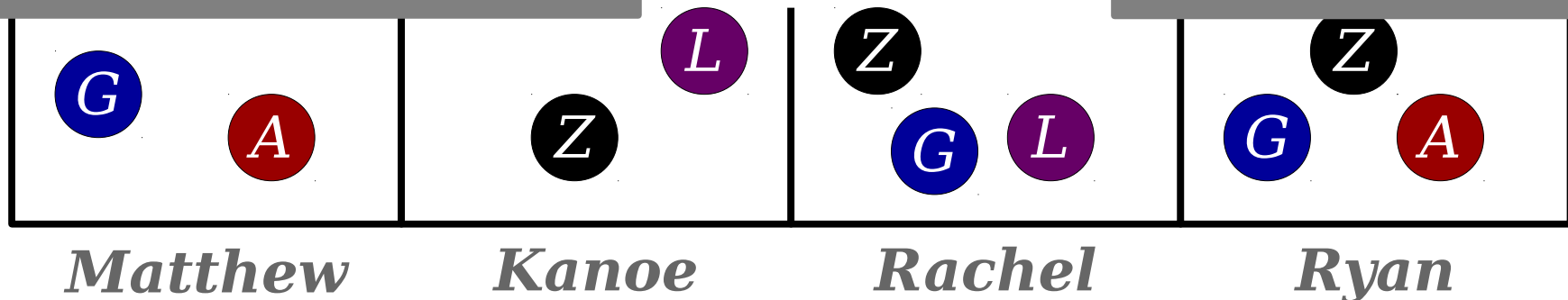
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- 90% of those people enjoyed **Get Out**,
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- 60% of those people enjoyed **Zootopia**.

No one enjoyed all four movies. How many people enjoyed at least one of *Get Out* and *Arrival*?”

Insight 1: Model movie preferences as balls (movies) in bins (people).

Insight 2: There are n total bins, one for each person.



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$$\begin{aligned} & .9n + .8n + .7n + .6n \\ & = 3n \end{aligned}$$

Insight 3: There are $3n$ balls being distributed into n bins.

Insight 4: The average number of balls in each bin is 3.

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Insight 5: No one enjoyed more than three movies...

Insight 6: ... so no one enjoyed fewer than three movies ...

Insight 7: ... so everyone enjoyed exactly three movies.

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Insight 8: You have to enjoy at least one of these movies to enjoy three of the four movies.

Conclusion: Everyone liked at least one of these two movies!

Theorem: In the scenario described here, all n people enjoyed at least one of *Get Out* and *Arrival*.

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Proof: Suppose there is a group of n people meeting these criteria.

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$$.9n + .8n + .7n + .6n = 3n,$$

and since there are n people, there are n bins.

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Now suppose for the sake of contradiction that someone didn't enjoy *Get Out* and didn't enjoy *Arrival*.

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We've reached a contradiction, so our assumption was wrong and each person enjoyed at least one of *Get Out* and *Arrival*. ■

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Going Further

- The pigeonhole principle can be used to prove a *ton* of amazing theorems. Here's a sampler:
 - There is always a way to fairly split rent among multiple people, even if different people want different rooms. (*Sperner's lemma*)
 - You and a friend can climb any mountain from two different starting points so that the two of you maintain the same altitude at each point in time. (*Mountain-climbing theorem*)
 - If you model coffee in a cup as a collection of infinitely many points and then stir the coffee, some point is always where it initially started. (*Brower's fixed-point theorem*)
 - A complex process that doesn't parallelize well must contain a large serial subprocess. (*Mirksy's theorem*)
 - Any positive integer n has a nonzero multiple that can be written purely using the digits 1 and 0. (*Doesn't have a name, but still cool!*)

More to Explore

- Interested in more about graphs and the pigeonhole principle? Check out...
 - ... **Math 107** (Graph Theory), a deep dive into graph theory.
 - ... **Math 108** (Combinatorics), which explores a bunch of results pertaining to graphs and counting things.
 - ... **CS161** (Algorithms), which explores algorithms for computing important properties of graphs.
 - ... **CS224W** (Deep Learning on Graphs), which uses a mix of mathematical and statistical techniques to explore graphs.

Next Time

- ***Midterm review session!***